

Nonlinear consolidation theory for prefabricated vertical drains with elliptic cylindrical assumption

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Abstract. Based on the influence of the non-linear change in the consolidation process of permeability coefficient and compressibility of soft soils and the theory of elliptic cylinders, derived the control equation of the nonlinear consolidation of the vertical foundation of the strip plastic drainage board. Further, according to the complete orthogonal space theory, the equivalent polynomial analytical expression of the nonlinear term is given. Then, the analytic solutions of the nonlinear consolidation are given, and through numerical calculation and verification analysis, it is considered that the Indrarantna (2005) approximation has a relatively small error at C_v/C_k (close to 1) and N_u (<4), but when C_v/C_k (difference from 1) and N_u are large (>4), the error is large.

1 Introduction

Based on the assumption that soil permeability and compressibility in the consolidation process are constant, Terzaghi [1] proposed a classical one-dimensional consolidation theory. However, the consolidation behavior of soft soil is complicated. Therefore, a series of researchers studied nonlinear consolidation theory. Davis & Raymond [2] firstly got an analytical solution for one-dimensional consolidation problem under instantaneous loading based on the assumption that the permeability coefficient k_v and the volume compression coefficient m_v changed synchronously. Soon after, Barden et al.[3] and Gibson [4] solved one-dimensional nonlinear consolidation theory with numerical methods based on different assumptions on the nonlinearity of soil unit. Mesri et al.[5] studied the one-dimensional large-strain nonlinear consolidation theory based on a series of simplification. Since then, many scholars have conducted in-depth researches from various aspects such as numerical analysis, laboratory tests and field tests. Lekha et.al [6] used approximate method solved a one-dimensional nonlinear consolidation.

As for the nonlinear consolidation of vertical drained foundation, Hansbo et.al [7] considered the non-linearity of the permeability coefficient in the consolidation process, analyzed exponential relationship between hydraulic gradient and penetration velocity. Indrarantna et al. [8] used the relationship between $e \sim \log \sigma'$ and $e \sim \log k_v$ to determine the relationship between the compression coefficient C_c and the horizontal permeability coefficient k_h , and then got an analytical solution for the consolidation of vertical drained ground.

Considering the real size of plastic vertical drain, Huang et al. [9] gained an analytical solution for the

consolidation of vertical drained ground with a assumption that the PVD shape is a elliptic cylinder instead of cylinder. Lu et al. [10] analyzed the nonlinear consolidation of vertical foundations with the approximate method.

Thereby, a nonlinear consolidation controlling equation based the elliptic cylinder assumption for vertical drain considering the non-linear change of permeability coefficient and compressibility of soft soil during the consolidation process is derived. Legendre polynomial space theory is applied in the following to solve this equation.

2 Description of physical model and assumptions

Because of the advantages of fast construction, good permeability and low cost, prefabricated vertical drain is most widely used among different drain materials in soft soil ground improvement. Fig. 1 is the equivalent calculation model for the elliptical cylinder of PVD [9]. In the figure, L is the calculation length of the drain; ρ_w is the length coordinates of the elliptic cylinder drain; ρ_s is the length coordinates of the confocal elliptic cylinder smear zone; ρ_e is the length coordinates of the confocal elliptic influence area of the drainage body; u is the excess pore water pressure in soil; k_w is the permeability coefficient of drain; k_h is the radial permeability coefficient of soil; k_s is the permeability coefficient of smear zone soil.

The basic assumptions are as follows:

(1) Equal strain assumption, the vertical deformation of the same depth between drain and surrounding soil layer are equal.

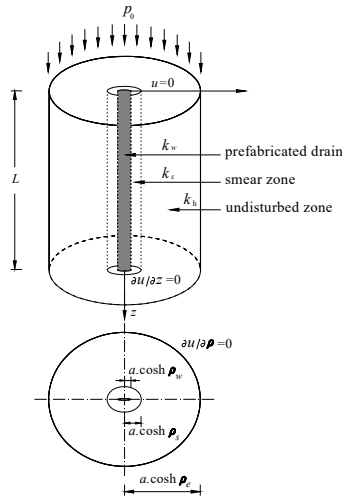
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(2) Only the normal flow in the ellipse unit is considered; the seepage in soil is consistent with Darcy's law; well resistance is ignored in drain well.

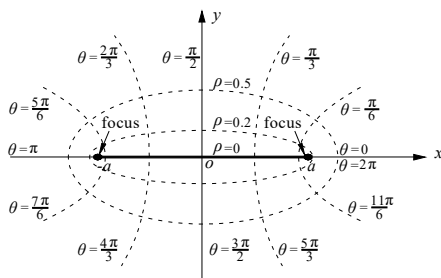
(3) The permeability coefficient of soil in the smear zone is changed synchronously with the permeability coefficient of the original soil area, k_s/k_h is a fixed value.

(4) The top surface is permeable, while the bottom surface and the periphery of the vertical drain foundation is impervious to water.

(5) The consolidation by radial flow and vertical flow can be considered separately according to Carrillo's law.



(a) Calculation model



(b) Elliptical coordinate system in ρ and θ coordinates

Fig.1. Diagram of unit cell with elliptic cylinder assumption on PVD

3 Basic equations and solution conditions

3.1 Constitutive relations

According to Mikasa^[5], the relationship between the void ratio e , permeability coefficient k_h and effective stress σ' is:

$$e = e_0 - C_c \lg \left(\frac{\sigma'}{\sigma_0'} \right) \quad (1)$$

$$e = e_0 - C_k \lg \left(\frac{k_h}{k_{h0}} \right) \quad (2)$$

where C_c and C_k are the compression index and penetration index of soil, respectively; e_0 and σ_0' are the initial void ratio and initial effective stress; k_{h0} is the initial permeability coefficient.

According to the effective stress principle, there is

$$\sigma' = \sigma_0' + p_0 - u_\rho \quad (3)$$

Assuming the distribution of additional stress is constant along the depth direction, then

$$\frac{m_{v0}}{m_v} = 1 + \frac{p_0}{\sigma_0'} - R_u \cdot \frac{p_0}{\sigma_0'} \quad (4)$$

where \bar{u}_ρ is the pore water pressure; $R_u = \bar{u}_\rho / p_0$.

Then it can be obtained that

$$\frac{k_{h(t)}}{k_{h0}} = \left(1 + \frac{p_0}{\sigma_0'} - R_u \cdot \frac{p_0}{\sigma_0'} \right)^{-C_c / C_k} \quad (5)$$

3.2 Balance equation and continuous equation

According to the theory of elliptical cylindrical coordinate system (Fig. 1), the basic foundation consolidation equation of the elliptic cylinder drainage body is obtained.

Balance equation:

$$\frac{\partial \varepsilon_v}{\partial t} = -m_v \frac{\partial u_\rho}{\partial t} \quad (6)$$

Continuous equations for the soil in smear zone and undisturbed area:

$$\begin{cases} \frac{k_s}{\gamma_w a^2 (\cosh^2 \rho - \cos^2 \theta)} \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{\partial \varepsilon_v}{\partial t}, & \rho_w \leq \rho \leq \rho_s \\ \frac{k_h}{\gamma_w a^2 (\cosh^2 \rho - \cos^2 \theta)} \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{\partial \varepsilon_v}{\partial t}, & \rho_s \leq \rho \leq \rho_e \end{cases} \quad (7)$$

where m_v is the volume compression coefficient; ε_v is the volume strain; u is the excess pore-water pressure; γ_w is the water bulk density; t is the consolidation time.

The above equation (6) and (7) are the basic consolidation equations in the elliptical cylindrical coordinate system.

3.3 Controlling equations

From the assumption of equal strain, ε_v is independent of ρ , and according to the biaxial symmetry of the ellipse and the excess pore pressure solution, it can be expressed as $u = f(\rho) + g(\rho) \cos 2\theta$. Substituting it into equation (7), then it can be obtained that

$$\left(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial \theta^2} \right) = \frac{1}{2} \left(\frac{\gamma_w a^2}{k_s} \frac{\partial \varepsilon_v}{\partial t} \right) (\cosh 2\rho + \cos 2\theta) \quad (8)$$

This ordinary differential equation can be solved by separating variables method as:

$$\begin{cases} f(\rho)'' = \frac{1}{2} \left(\frac{\gamma_w a^2}{k_s} \frac{\partial \varepsilon_v}{\partial t} \right) \cosh 2\rho \\ g(\rho)'' - 4g(\rho) = \frac{1}{2} \left(\frac{\gamma_w a^2}{k_s} \frac{\partial \varepsilon_v}{\partial t} \right) \end{cases} \quad (9)$$

According to the boundary conditions: $\frac{\partial f}{\partial \rho} \Big|_{\rho=\rho_e} = 0$, $\frac{\partial g}{\partial \rho} \Big|_{\rho=\rho_e} = 0$, $g \Big|_{\rho=\rho_w} = 0$, and continuous equations, there are:

$f_{ws} \Big|_{\rho=\rho_s} = f_{se} \Big|_{\rho=\rho_s}$, $g_{ws} \Big|_{\rho=\rho_s} = g_{se} \Big|_{\rho=\rho_s}$, the excess pore water pressure in the smear zone and the nature soil area can be solved, as shown in Huang et al. ^[9].

According to the principle of integral average, the average excess pore pressure at any depth in the foundation can be expressed. Then it can be calculated as follows

$$\bar{u}_\rho = \frac{\gamma_w (\cosh \rho_e)^2}{2k_{h(t)}} a^2 F_h \times \left(\frac{\partial \varepsilon_v}{\partial t} \right) \quad (10)$$

Equation (10) is the deduced controlling equation that ignored well resistance of PVD with elliptical cylindrical assumption. Parameter F_h can be calculated by the basic parameters $\rho_w, \rho_s, \rho_c, a$, and k_h/k_s . The calculation process is attached in Huang et al. [9].

Substituting equation (5), (6) into equation (10), and combining $R_u = \frac{\bar{u}_\rho}{P_0}$, it can be obtained that

$$R_u = -\frac{\gamma_w a^2 (\cosh \rho_e)^2 m_{v0}}{2 k_{h0}} F_h \times \left(1 + \frac{p_0}{\sigma_0'} - R_u \cdot \frac{p_0}{\sigma_0'} \right)^{C_c/C_k - 1} \frac{\partial R_u}{\partial t} \quad (11)$$

Assume a dimensionless time parameter :

$$T = \frac{c_{h0}}{4a^2 (\cosh \rho_e)^2} t, \quad N_u = \frac{p_0}{\sigma_0'}$$

can be simplified as a dimensionless equation as

$$\frac{\partial R_u}{\partial T} = -\frac{8}{F_h} (1 + N_u - R_u \cdot N_u)^{1 - C_c/C_k} \cdot R_u \quad (12)$$

where

$$c_{h0} = \frac{k_{h0} (1 + e_0)}{\gamma_w \left(\frac{\partial e}{\partial \sigma'} \right)_{e=0}}, \quad m_{v0} = \left(\frac{\partial e}{\partial \sigma'} \right)_{e=0} \cdot \frac{1}{1 + e_0}$$

The above equation (12) is the basic equation for the nonlinear consolidation of the PVD foundation in the elliptical cylindrical coordinate system without considering the well resistance.

4 Nonlinear equation solving

According to the nonlinear control equation (12) can be transformed into:

$$-\frac{8}{F_h} \partial T = \frac{(1 + N_u - R_u \cdot N_u)^{C_c/C_k - 1}}{R_u} \partial R_u \quad (13)$$

In this paper, the rigorous analytical solution of equation (13) is given based on Hillbert's complete orthogonal space theory in functional analysis. Let $C_c / C_k = \langle C_c / C_k \rangle + M_{ck}$, and $\langle C_c / C_k \rangle, M_{ck}$ are functions of C_c / C_k , and their values are as follows:

while $0 < C_c / C_k \leq 2$, $\begin{cases} \langle C_c / C_k \rangle = C_c / C_k; \\ M_{ck} = 0 \end{cases}$ while

$2 < C_c / C_k \leq 3$, $\begin{cases} \langle C_c / C_k \rangle = C_c / C_k - 1; \\ M_{ck} = 1 \end{cases}$

In turn, nonlinear equation (13) can be transformed into:

$$-\frac{8}{F_h} \partial T = (1 + N_u - R_u \cdot N_u)^{\langle C_c / C_k \rangle - 1} \times \frac{(1 + N_u - R_u \cdot N_u)^{M_{ck}}}{R_u} \partial R_u \quad (14)$$

The non-linear item on the right side of (19) is: $G(R_u) = (1 + N_u - R_u \cdot N_u)^{\langle C_c / C_k \rangle - 1}$, when $0 < R_u < 1$, The function $G(R_u)$ about R_u is extended from the interval

[0,1] to the even function in the complete Legendre polynomial space [-1,1].

According to Hillbert orthogonal complete space theory, the Negejrond polynomial orthogonal can be represented by the Legendre polynomial equivalent:

$$G(R_u) = (1 + N_u - R_u \cdot N_u)^{\langle C_c / C_k \rangle - 1} = a_0 P_0(x) + a_2 P_2(x) + a_4 P_4(x) + \dots \quad (15)$$

Where $P_0(x), P_2(x)$, and $P_4(x)$ represent Legendre polynomials, respectively, $P_0(x) = 1, P_2(x) = \frac{1}{2}(3x^2 - 1),$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

Then equation (14) can in turn be transformed into:

$$-\frac{8}{F_h} \partial T = \frac{a_0 P_0(x) + a_2 P_2(x) + a_4 P_4(x)}{R_u} (1 + N_u - R_u \cdot N_u)^{M_{ck}} \partial R_u \quad (16)$$

According to the orthogonal inner product relationship, it can be seen that the formulas a_0, a_2, a_4 can be given as:

$$a_0 = \frac{\int_0^1 G(x) \times P_0(x) dx}{\int_0^1 P_0(x) \times P_0(x) dx},$$

$$a_2 = \frac{\int_0^1 G(x) \times P_2(x) dx}{\int_0^1 P_2(x) \times P_2(x) dx}, \quad a_4 = \frac{\int_0^1 G(x) \times P_4(x) dx}{\int_0^1 P_4(x) \times P_4(x) dx} \quad (17)$$

Then we can give the analytical expressions of the parameters a_0, a_2, a_4 about the constant parameters $N_u = \frac{p_0}{\sigma_0'}, C_c / C_k$:

$$a_0 = \frac{y_0}{N_u},$$

$$a_2 = \frac{30}{4N_u^3} \left\{ y_2 - 2(1 + N_u) \cdot y_1 + (1 + N_u)^2 \cdot y_0 \right\} - \frac{10}{4N_u} y_0,$$

$$a_4 = \frac{9}{8} \left\{ \frac{30}{N_u^3} \left[y_2 - 2(1 + N_u) y_1 + (1 + N_u)^2 y_0 \right] + \frac{3}{N_u} y_0 \right\} \quad (18)$$

$$+ \frac{315}{8N_u^5} \left[y_4 - 4(1 + N_u) y_3 + 6(1 + N_u)^2 y_2 - 4(1 + N_u)^3 y_1 + (1 + N_u)^4 y_0 \right]$$

where:

$$y_0 = \frac{(1 + N_u)^{\langle C_c / C_k \rangle} - 1}{\langle C_c / C_k \rangle}; \quad y_1 = \frac{(1 + N_u)^{\langle C_c / C_k \rangle + 1} - 1}{\langle C_c / C_k \rangle + 1};$$

$$y_2 = \frac{(1 + N_u)^{\langle C_c / C_k \rangle + 2} - 1}{\langle C_c / C_k \rangle + 2}; \quad y_3 = \frac{(1 + N_u)^{\langle C_c / C_k \rangle + 3} - 1}{\langle C_c / C_k \rangle + 3};$$

$$y_4 = \frac{(1 + N_u)^{\langle C_c / C_k \rangle + 4} - 1}{\langle C_c / C_k \rangle + 4}; \quad (19)$$

The Legendre polynomials $P_0(x), P_2(x)$, and $P_4(x)$ are substituted into equation (16). The following simple equations can be obtained by simple changes:

$$-\frac{8}{F_h} \partial T = (b_0 + b_2 R_u^2 + b_4 R_u^4) \times \frac{(1 + N_u - R_u \cdot N_u)^{M_{ck}}}{R_u} \partial R_u \quad (20)$$

where : $b_0 = a_0 - \frac{1}{2} a_2 + \frac{3}{8} a_4; \quad b_2 = \frac{3}{2} a_2 - \frac{30}{8} a_4;$

$$b_4 = \frac{35}{8} a_4.$$

According to equation (20) deduced in this paper, we can give an analytical solution, as follows

$$1) \text{ while } 0 < C_c / C_k \leq 2, \quad \begin{cases} \langle C_c / C_k \rangle = C_c / C_k \\ M_{ck} = 0 \end{cases},$$

according to the separation integral, and the boundary conditions $T = 0$ and $R_u = 1$,

$$T = -\frac{F_h}{8} \left\{ b_0 \ln R_u + \frac{b_2}{2} (R_u^2 - 1) + \frac{b_4}{4} (R_u^4 - 1) \right\} \quad (21)$$

$$2) \text{ while } 2 < C_c / C_k \leq 3,$$

$\langle C_c / C_k \rangle = C_c / C_k - 1$, according to the boundary $M_{ck} = 1$

conditions $T = 0$ and $R_u = 1$,

$$T = -\frac{F_h}{8} \left\{ \begin{aligned} &(1 + N_u) b_0 \ln R_u - N_u b_0 (R_u - 1) + (1 + N_u) \frac{b_2}{2} (R_u^2 - 1) \\ &- N_u \frac{b_2}{3} (R_u^3 - 1) + (1 + N_u) \frac{b_4}{4} (R_u^4 - 1) - N_u \frac{b_4}{5} (R_u^5 - 1) \end{aligned} \right\} \quad (22)$$

Among them, total average degree of consolidation $U=1-R_u$. So far, the equivalent analytical solution of the nonlinear consolidation of shaft foundations has been given, that is, the relationship between the degree of consolidation, the dimensionless time factor T , and the analytical solution.

5 Verification and discussion

5.1 Verification

In order to compare and analyze the relationship between the nonlinear consolidation analytical solution (22) in this paper and the nonlinear consolidation approximate solution in B Indraratna (2005) under ideal shaft conditions, this paper compares and analyzes the following calculation conditions:

- Condition (a) : $C_c/C_k=1, N_u=20, F_h=8$;
- Condition (b) : $C_c/C_k=0.2, N_u=10, F_h=8$;
- Condition (c) : $C_c/C_k=0.2, N_u=20, F_h=8$;
- Condition (d) : $C_c/C_k=1.95, N_u=10, F_h=8$;
- Condition (e) : $C_c/C_k=1.95, N_u=20, F_h=8$;
- Condition (f) : $C_c/C_k=2.95, N_u=20, F_h=8$;

It can be seen from Fig.2 that when $C_c/C_k \rightarrow 1$, the analytical solution of nonlinear consolidation in this paper degenerates to classic linear consolidation. At this time, the analytical solution in this paper is consistent with the calculation results of Indraratna et al. [8] approximation method, and is consistent with the accurate numerical solutions.

5.2 Discussion

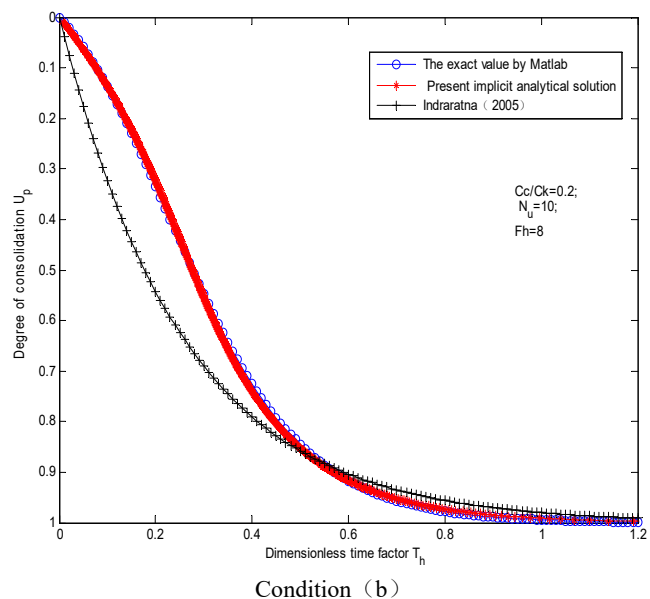
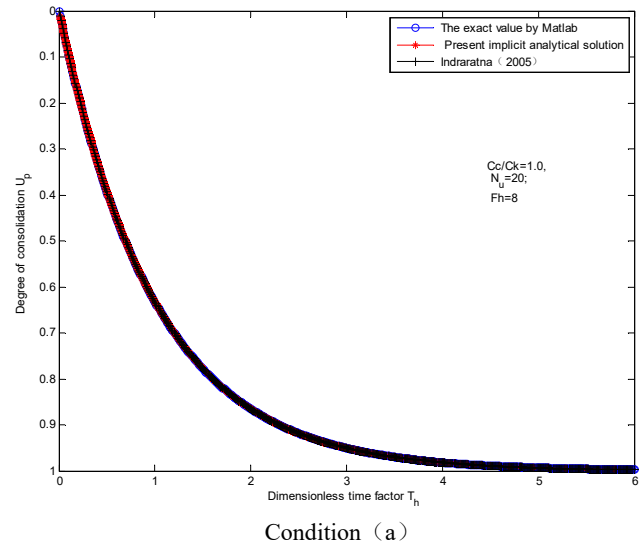
Through the comparison of numerical calculations under the above different conditions:

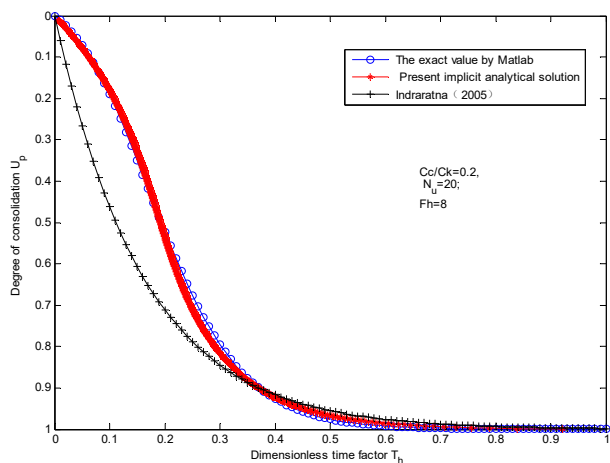
(1) The analytical solution in this paper degenerates into an analytical solution similar to an ideal shaft when c_c/c_k approaches 1, which is consistent with the Indraratna [8] approximation, but without the dimensionless parameters F_h , the main reason is that the

difference between the shape of the strip-shaped plastic drainage plate and the circular sand well is larger.

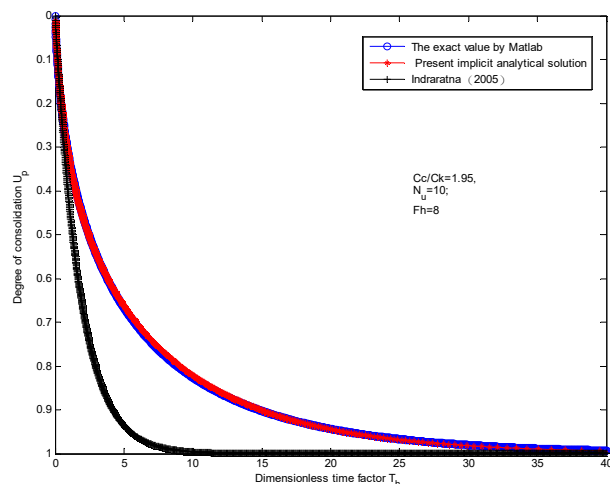
(2) The larger the relative value of the pre-loading force N_u is ($N_u=10$), the deviation of Indraratna [8] approximation and the exact numerical solution is larger even the degree of consolidation error as high as 30% when C_c/C_k deviates from 1 and gradually approaches 0.2, The results of the special Indraratna [8] approximation seriously overestimate the nonlinear consolidation rate of shaft foundations when the degree of consolidation is less than 90%. In contrast, the results of the analytical solution in this paper are basically the same to the exact numerical solution.

(3) The larger the relative value of the preloading force N_u is ($N_u=10$), the deviation of Indraratna [8] approximation and the exact numerical solution is larger even the degree of consolidation error as high as 40% when C_c/C_k deviates from 1 and gradually approaches 2. The results of the Indraratna [8] approximation seriously overestimates the nonlinear consolidation rate of the shaft foundation especially when the degree of consolidation is less than 95%. In contrast, the results of the analytical solution in this paper are basically the same to the exact numerical solution.

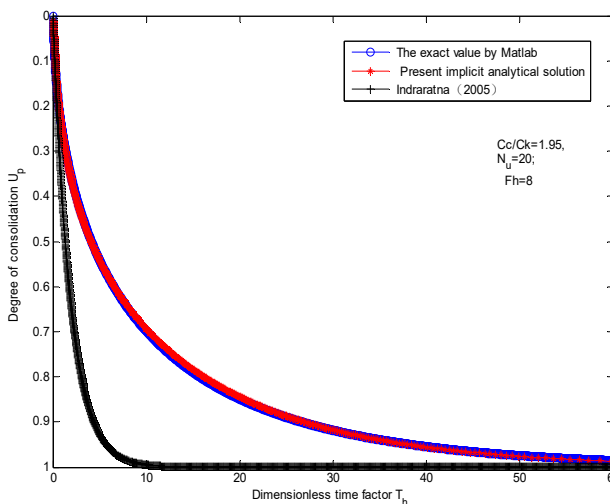




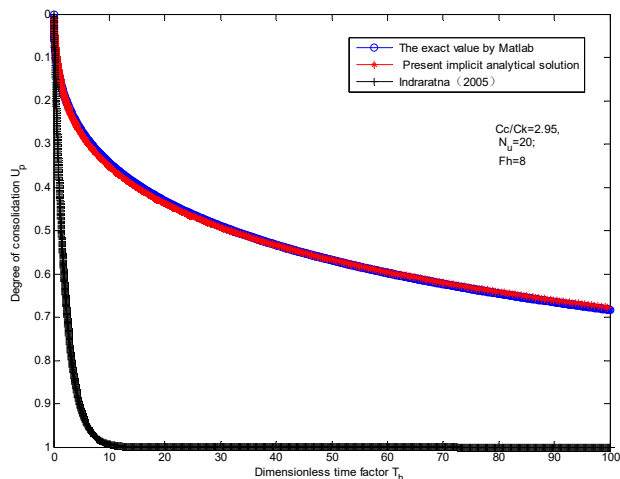
Condition (c)



Condition (d)



Condition (e)



Condition (f)

Fig. 2 Comparison of analytical solutions and different solutions

6 Conclusion

Firstly, based on the theory of elliptic cylinders, considering the nonlinear influence of permeability coefficient and compressibility of soft soil in the consolidation process, the the nonlinear consolidation of shaft foundation is discussed.

Secondly, derived the control equation of the nonlinear consolidation of the vertical foundation of the strip plastic drainage board. Further, according to the complete orthogonal space theory, the equivalent polynomial analytical expression of the nonlinear term is given. Then, the analytic solution of the nonlinear consolidation degree are given respectively.

Thirdly, through numerical calculation and verification analysis, it is considered that the Indraratna (2005) approximation has a relatively small error at C_c/C_k (close to 1) and N_u (<4), but when C_c/C_k (difference from 1) and N_u are large (>4), the error is large .

Finally, it is recommended to conduct follow-up in-depth research on the nonlinear consolidation of the shaft foundation under the influence of well resistance.

Acknowledgments

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