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Optimal modeling of the heat transfer of a viscous incompressible liquid

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Abstract. This paper discusses mathematical models of the heat transfer process of a viscous incompressible fluid. Optimal control methods are used to solve the problem of optimal modeling. Questions of linearization of the Navier - Stokes equation are considered. The optimal modes (optimal functional dependencies) of the pump and heating device are found depending on the fluid flow rate.

Introduction

The development of technological solutions for the use of heat pipes and heat pumps in heat supply systems is one of the most important scientific and practical problems of our time. As can be seen from the literature [1-6], there is an increasing interest in the development of technological solutions for the use of pipelines in heat supply systems, modeling the corresponding technological processes and determining the main physical parameters. Therefore, at present, the study of them both from the theoretical and practical sides is one of the urgent tasks. As is known, in the study of technological processes of heat transfer, gas transmission and power transmission, the established fundamental laws of physics and their basic equations written according to the similarity condition are used. In scientific and practical research, studying some individual processes, either theoretically or experimentally, results are obtained in relation to certain physical parameters. In similar engineering-physical objects-processes (for example, thermal, gas and electrical) based on the laws of physics, the differential equations of Navier-Stokes or Maxwell are used. By simplifying these equations, a relationship is established between the parameters; come to a conclusion about physical processes. Of course, the results obtained from these simplifications are very practical and valuable. However, guidelines that are given without taking into account the most important physical parameters are local in nature, lead to large discrepancies between theoretical results and actual processes, and, rather, the results refer to the functional relationship of the process, which does not fully describe its actual steady state. Since the second half of the last century, in the scientific research and experiments cited, they have evaded the option of adding additional terms for the Navier-Stokes equations in accordance with the state of real technological processes. For example, in the scientific monograph [4] it is shown that the Navier-Stokes equations can have solutions that do not correspond to the Reynolds number in infinitely distant regions (pipes), and differential equations are studied with the addition of an additional linear term, although, as noted, it is not experimentally reasonable.

Optimization of engineering and physical processes using nonlinear Navier-Stokes equations is one of the poorly studied problems. The study of inhomogeneous or non-self-adjoint boundary conditions, or the effect of pumps in heat transfer networks, or the addition of an additional term to the Navier-Stokes equations, more suitable for technological processes of heat conduction, is one of the urgent problems. In modern installations, pumps are widely used in the heat transfer process, which affect the rate of heat flow, which leads to convective heat transfer in a turbulent heat flow. As a result, automation and finding the optimal pump operation mode depending on the speed, temperature and time of the heat flow during the heat transfer process is one of the most modern technological problems.

In this work, for a complete study of engineering and technological processes, i.e. for real processes of optimal control synthesis, we propose a systematic approach - an optimal model for adding the functional dependence of control functions on the fluid flow rate as an additional term to the Navier-Stokes equations.

1 Statement of the optimization problem for the differential Navier-Stokes equations

For a nonstationary and nonlinear vector differential Navier-Stokes equation, consider the initial boundary conditions (the density is assumed to be ρ = 1) [3-6]:

$$\mathcal{L}w \equiv w_t - \nu \Delta w + \alpha v_k w_{x_k} = -grad p + f + f_1,$$

$$div w = 0 \qquad (1)$$

$$w|_{t=0} = a(x), \quad div a = 0, \quad w|_{S_T} = g(t), \qquad (2)$$

where α is positive number, $t \in [0, T], \Omega$ is a region (or body) of three-dimensional Cartesian space with a

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sufficiently smooth surface S, $Q_T = \Omega X(0,T)$, $S_T = SX[0,T]$; Δ is a Laplace operator and ν is a constant coefficient of viscosity. Let`s denote by $L_2(Q_T)$ (and $L_2(\Omega)$) as the set (space) of three-dimensional vector functions consisting of component-functions that are square-integrable in the domain Q_T (or Ω). We introduce in this space the usual scalar product and the norm of elements. It is assumed that in the equation (1), $p = p(x,t)$ is a pressure of exterior forces $f = f(x,t)$, $f_1 = f_1(x,t)$, and in the conditions of (2), the functions $g = g(t)$, $a(x)$ possess the necessary differential properties, whereas $w(x,t) = v_1(x,t)(, v_2(x,t), \quad v_3(x,t))$ $(x = (x_1, x_2, x_3))$ is vector-rate of the fluid, in which $w_t(x,t)$ characterizes the (local) rate of change by time at points of x. In the equation (1), nonlinear vector $v_k w_{x_k}$ characterizes the change of rate (or velocity) from one point to another point. If $\alpha = 0$, then we will get the linear nan-stationary Navier-Stokes equation [4-6]. In the equation (1), f_1 is constant force. In the following optimal modeling issue, boundary conditions of other types (for example, non-self-adjoint boundary conditions of the Bitsadze-Samarskii type [7]) are considered similarly, with the difference that in order to apply the method of spectral decomposition of the solution of a boundary value problem, it is necessary to use the apparatus of the theory of non-self-adjoint operators [7]. The existence and uniqueness of the solution to the problems (1), (2) have been proven in [4-6].

In boundary conditions (2), the vector function g (t) describes the action of the pump in the boundary mode, and the vector function f characterizes the difference between the surface temperature of the heating device and the temperature of the liquid (lift force).

Under conditions (1), (2), the problem of optimal modeling of the heat transfer process is formulated as follows. Find the control functions *g(t), f(x, t)* as functions of the fluid velocity, i.e. find the synthesizing *f = f (w, t), g = g (w, t)* control functions that depend on the velocity vector *w* and ensure that the velocity of the controlled fluid flow approaches the specified normal velocity *φ (x , t)* of the liquid, and at the end of the controlled process also approached the specified normal speed *ψ (x)* and, so that the energies of forces (pump, heating device) acting on convective heat transfer were minimal.

Then, the criteria of optimal modeling problem can be written as follows:

$$I[t_0, g, f] = \alpha_1 \int_{t_0}^{T} \|w - \varphi\|^2\, dt + +\alpha_2 \|w_T - \psi\|^2 \int_{t_0}^{T} (\alpha_3 \|f\|^2 dt + \alpha_4 |g(t)|^2) dt, \qquad (3)$$

where $t_0 = 0$, $w_T = w(x,T)$, $\alpha_i (i = 1,2,3,4)$ are the given positive numbers. The end (at point T) of controlling process is fixed. In this way, from the task of optimal modeling it can be concluded up that in order to find the synthesizing functions $f = f(w,t)$ and $g = g(w,t)$ of control, which depend on flow rate and, together with the corresponding solution w to the initial-boundary value problems (1), (2), the functional (3) minimum value would be assigned.

2 Solving the problem of optimal modeling

To find a solution to the synthesis problems (1), (2), (3), the dynamic programming method was used [7]. According to this method, the minimum function is denoted by $S[t,w]$. As a result, $S[t_0, w(t,.)] = \min_{g,f} I[t_0, g, f]$. In accordance with the dynamic programming method, a nonlinear functional equation of R. Bellman [7] is obtained as follows:

$$-\frac{\partial S}{\partial t} = \min_{p,f} \{ \nu(w, \Delta u) + \alpha(w, v_k u_{x_k}) +$$

$$+(-gradp + f + f_1, u) - \qquad (4)$$

$$-\nu g(t) u_{xS_T} + \alpha_1 \|w - \varphi\|^2 + \alpha_3 \|f\|^2 + \alpha_4 |g(t)|^2 \},$$

$$S[T, w(T,x)] = \alpha_2 \|w_T - \psi\|^2, \ u_{S_T} = u|_{S_T} = 0, \quad (5)$$

where $u = u(t,w)$ is the Fréchet functional derivative of the Bellman function $S[t,w]$. From the equation (4) we find the control functions

$$f(x,t) = -\frac{1}{2\alpha_3} u(t,w); \ g(t) = \frac{1}{2\alpha_4} \nu u_{xS_T}(t,w). \quad (6)$$

The control functions found by formulas (6) are synthesizing control functions, that is, as required in the optimal modeling problem formulated above, they are functions that depend on the fluid flow rate. Substituting them in (1), (2), we obtain equations with initial boundary conditions describing the rate of the optimal fluid flow.

3 Solution of the Bellman equation

If the fluid flow is described by a linear equation, then in equations (1), (4) and (6) it is necessary to formally set $\alpha = 0$. In this case, the Bellman function, which is determined by using equations (4) - (6), is should be the sum of square and linear forms and, therefore, the functional derivative $u(t,w)$ is expressed by the linear form of $R(t)w$, moreover, matrix operator $R(t)$ is determined through the nonlinear matrix of Riccati equations [7-11]. Indeed, if the solution to problem (4), (5) is sought in the form:

$$S[t, w(\cdot,t)] = (R(t)w, w) + (k(t), w) + \eta(t),$$

where $k(t)$ is a three-dimensional vector function, $\eta(t)$ is a scalar function, the operator matrix $R(t)$ is symmetric and is defined as follows: $R(t) = R(t,x,y) \equiv R(t,y,x)$, then calculating the Frechet and t derivatives of the functional S we find:

$$\partial S / \partial t = (R'(t)w, w) + (k'(t), w) + \eta'(t),$$

$$u(t,w) = 2R(t)w + k(t).$$

Accordingly,$\Delta u = \Delta(2R(t)w + k(t)) = 2\Delta R(t)w + \Delta k(t)$. To determine the matrix operator $R(t)$, the vector $k(t)$, and the scalar function $\eta(t)$, such system of operator equations are obtained:

$$R'(t)+2\nu\Delta R(t)-\frac{1}{\alpha_3}R^2(t)-$$

$$-\frac{\nu^2}{\alpha_4}R^2_{xS_T}(t)+\alpha_1\delta=0, \qquad (7)$$

$$k'(t)+\nu\Delta k(t)-\frac{1}{\alpha_3}R(t)k(t)-\frac{\nu^2}{\alpha_4}R_{xS_T}(t)k_{xS_T}(t)-$$

$$-2\alpha_1\varphi(t)+2R(t)f_1=0;$$

$$\eta'^{(t)}-\frac{1}{4\alpha_3}\|k(t)\|^2-\frac{\nu^2}{4\alpha_4}\left\|k_{xS_T}(t)\right\|^2+(f_1,k(t))+$$

$$+\alpha_1\|\varphi(t)\|^2=0$$

with the following initial-boundary conditions:

$$R(T)=\alpha_2\delta, k(T)=-2\alpha_2\psi, \quad \eta(T)=\alpha_2\|\psi\|^2,$$

$$R_{S_T}(t)=k_{S_T}(t)=0,$$

where δ is a Dirac delta function. The work [7] is devoted to the solvability of the obtained system of equations.

And in the case when $\alpha\neq 0$, the Bellman functional can be sought (as in the linear case) as a sum of square and linear forms. But in the latter case, a nonlinear term with the coefficient α is added to the matrix Riccati equation. In the general case, to find the value of the functional *S*, and hence the value of the control functions from relations (6), one can use the approximate method shown in [7]. According to this method, the time interval of the control process is divided into elementary parts and the initial condition (5) is used; first, a solution is found for an elementary section containing the initial condition (5), then a solution is found step by step for the entire control interval.

Remark. If the viscosity coefficient $\nu\to 0$, then the effect of the pump tends to be zero. In this case, $g(t)=0$ and the optimal state of the liquid is described by the equation (1) without the second term with ν, provided that $f(x,t)=-\frac{1}{2\alpha_3}u(t,w)$ and the boundary conditions are considered to be as follows:

$$w|_{t=0}=a(x),\ diva=0,\ \ w|_{S_T}=0.$$

4 Linearization of the Novier-Stokes equations

Note that equations (1) and (7) are nonlinear, i.e. quasilinear equations. Since synthesizing optimal controls are expressed in terms of linear forms $u(t,w)$ i.e. is expressed by the linear form $R(t)w$,, i.e. using the solution *R(t)* of the matrix Riccati equation, and this nonlinear equation is further complicated by the fact that for $\alpha\neq 0$ it contains a nonlinear term with v_k then an important modeling problem is the problem of linearization of the control process. In the literature [4], to linearize equation (1), it is proposed to replace v_k with another given b (x) (as noted there, by unconfirmed experience) vector. In this paper, we propose to replace v_k for $\alpha\neq 0$, either with the corresponding components of the solution (state vector) $w(x,t)$, obtained by the optimal control for $\alpha=0$, or replace the solution to the initial-boundary value problem (1) - (3) for $f=g=0$, or its approximate solution.

5 Other problems of optimal modeling

The optimal model through equations (1) - (6) is written for a fixed state of the last moment of the control process. If the end T of the control process is not fixed, then in the Bellman equation the $S[t,w]$ does not depend on the t time parameter, in other words, $S[t,w]\equiv S[w]$. As a result, in the equation (4), setting $\partial S/\partial t=0$, for $S[w]$ we will obtain the functional equation. Therefore, to determine the matrix operator R, corresponding algebraic matrix equation of Riccati [7] will be obtained. In this case, solving the problem is simplified compared to a dynamic situation. The optimal modeling problem is also solved for the cases when $T=\infty$. In this case, by similarity, the optimal fluid flow regime is proposed. In order to write an optimal stationary external model for stationary Navier-Stokes equations, it is necessary to add an additional condition which is $w|_{|x|\to\infty}=w^\infty$ and determine the minimum of the corresponding energy function. Similarly, it is proposed to use the maximum principle [8-11] in order to solve the problem of optimal modeling of a stationary heat transfer process in cylindrical coordinates, when the axis of a circular straight pipe is directed symmetrically along the x coordinate axis.

If the heat transfer regime is based on non-uniform and non-self-adjoint boundary conditions, for example, of the Bitsadze-Samarskii type, in which the one-dimensional case can be written as $w(0,t)=0$; $w_x(0,t)-w_x(l,t)=g(t)$, then in order to solve the boundary value problems, it is proposed to apply the method of spectral expansion in the Riesz basis of root vectors [7, 12].

6 Simplified Navier-Stokes equations

The literature contains numerous examples of the application of the Navier-Stokes equations (see bibl.). The new mathematical models obtained in this work make it possible to take into account the influence of external forces depending on the fluid flow rate. Naturally, in order to apply the results obtained, the proposed optimal mathematical models must be appropriately simplified. One of these simplifications can be given for the cases of two-dimensional and one-dimensional fluid flow. For example, in a one-dimensional flow, the linear Navier-Stokes equation with an additional found optimal term has the form ($\alpha=0$)

$$\frac{\partial v_1}{\partial t}-\nu\frac{\partial^2 v_1}{\partial x^2}=-\frac{dp}{dx}-\frac{1}{2\alpha_3}\int_0^l R(t,x,y)v_1(t,y)-$$

$$-\frac{1}{2\alpha_3}k(t,x)++f_1(t,x). \quad (8)$$

This equation should be solved with the initial and boundary conditions corresponding to (2):

$$v_1|_{S_T}=\frac{\nu}{2\alpha_4}\left(2\int_0^l R_{xS_T}(t,x,y)v_1(t,y)dy+k_{xS_T}(t,x)\right);$$
$$v_1(x,0)=a=const.$$

Equation (8) is a linear integro-differential equation in partial derivatives with respect to $v_1=v_1(x,t)$. The integrand function $R(t,x,y)$ is a positive solution of the Riccati equation and it is the stabilizing integral kernel of the fluid velocity, which is very important in the process of heat transfer in general and the reliability of the model power system under consideration; $k(t,x)$- the solution of the linear equation (7) takes into account the influence of a constantly acting force $f_1(x,t)$, also from the experience of the accepted-desired function-velocity $\varphi(x,t)$. A constantly acting force can characterize the influence of geometric parameters on the flow rate, for example, it can take into account the effect of a non-circular cross section in bent or rough pipes. It should be noted that the optimal practical calculations of the operating mode of the heating device and pump should be carried out from the optimal law of synthesizing functions of controls:

$$f(x,t;v_1)=-\frac{1}{2\alpha_3}u(t,v_1)=$$
$$-\frac{1}{2\alpha_3}\left(2\int_0^l R(t,x,y)v_1(t,y)dy+k(t,x)\right);$$
$$g(t;v_1)=\frac{1}{2\alpha_4}\nu u_{xS_T}(t,w)=$$
$$\frac{\nu}{2\alpha_4}\left(2\int_0^l R_{xS_T}(t,x,y)v_1(t,y)dy+k_{xS_T}(t,x)\right).$$

Thus, the proposed mathematical model of the optimal fluid velocity differs significantly from the traditionally used mathematical model of the heat transfer process [13] and will be useful in practical calculations of the reliability of power systems. In this case, we note that if the end of the time interval of the heat transfer process is not fixed, then in equation (8) it is necessary to set $\frac{\partial v_1}{\partial t}=0$ and this equation is simplified in this case.

In two-dimensional or three-dimensional problems with respect to the components of the optimal velocity vector, a system of two or three nonlinear integro-differential equations in partial derivatives is obtained, respectively. These equations will be useful in refining the values of the optimal heat transfer parameters, for example, the boundary layer of a fluid flow, which is an important technical aspect of the reliability of power systems.

Conclusion

The proposed optimal control methods for studying the heat transfer process, solving problems of optimal modeling, as well as the introduced stabilizing additional terms in the system of Navier-Stokes equations, the obtained optimal functional dependences of the heating device and pump will be useful in the study and analysis of practically required problems. The simplified mathematical models indicated in the work can be widely used in calculations and refinement of physical parameters, which is an important point in technology and the optimal design of reliable energy systems in general.

In this regard, in theoretical and practical studies, it is advisable to proceed from the proposed theoretical results and apply the found optimal model of the heat transfer process (the system of Navier-Stokes equations with stabilizing additional terms) and the optimal functional dependences of the pump and heating device mode on the fluid flow rate.

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