Coupled System of Boundary Value Problems by Galerkin Method with Cubic B-Splines

Kasi Viswanadham K.N.S.

Department of Mathematics, National Institute of Technology, Warangal – 506 004, India

Abstract. Coupled system of second order linear and nonlinear boundary value problems occur in various fields of Science and Engineering including heat and mass transfer. In the formulation of the problem, any one of 81 possible types of boundary conditions may occur. These 81 possible boundary conditions are written as a combination of four boundary conditions. To solve a coupled system of boundary value problem with these converted boundary conditions, a Galerkin method with cubic B-splines as basis functions has been developed. The basis functions have been redefined into a new set of basis functions which vanish on the boundary. The nonlinear boundary value problems are solved with the help of quasilinearization technique. Several linear and nonlinear boundary value problems are presented to test the efficiency of the proposed method and found that numerical results obtained by the present method are in good agreement with the exact solutions available in the literature.

1 Introduction

It is well known that, many problems in the areas of science and engineering are modeled by second order ordinary differential systems [1-3]. There are only few methods available to solve the coupled system of linear or nonlinear differential equations. In this paper we considered a system of second order linear boundary value problems of the type

$$\begin{aligned} &a_0(x)u_1'' + a_1(x)u_1' + a_2(x)u_1 \\ &+a_3(x)u_2'' + a_4(x)u_2' + a_5(x)u_2 = f_1(x) \end{aligned} \qquad a < x < b \quad (1.1)$$

$$\begin{aligned} &b_0(x)u_1'' + b_1(x)u_1' + b_2(x)u_1 \\ &+b_3(x)u_2'' + b_4(x)u_2' + b_5(x)u_2 = f_2(x) \end{aligned} \qquad a < x < b \quad (1.2)$$

subject to the boundary conditions

$$\sigma_1 u_1'(a) + \sigma_2 u_1(a) = u_{10} \qquad (1.3a)$$

$$\sigma_3 u_1'(b) + \sigma_4 u_1(b) = u_{11} \qquad (1.3b)$$

$$\mu_1 u_2'(a) + \mu_2 u_2(a) = u_{20} \qquad (1.3c)$$

$$\mu_3 u_2'(b) + \mu_4 u_2(b) = u_{21} \qquad (1.3d)$$

where σ_1, σ_2, σ_3, σ_4, μ_1, μ_2, μ_3, μ_4, u_{10}, u_{11}, u_{20}, u_{21} are finite real constants and $a_0(x)$, $a_1(x)$,…, $a_5(x)$, $b_0(x)$, $b_1(x)$, …, $b_5(x)$, $f_1(x)$, $f_2(x)$ are all continuous on $[a,b]$.

In many of applications we get a system of second order boundary value problems with both linear and nonlinear type. Here we considered a linear system of second order boundary value problems (1.1)-(1.2) along with 81 possible types of boundary conditions. If there is a coupled nonlinear system of second order boundary value problems, it can be converted into a sequence of coupled system of linear second order boundary value problems by using quasilinearization technique [4]. The limit of solutions of these generated linear boundary value problems is the solution of the nonlinear boundary value problem. Geng & Cui [5] and Dehghan & Abbas [6] solved coupled linear and coupled nonlinear system of boundary value problems with homogeneous boundary conditions only. In [6] they presented a sinc collocation method to solve the system. Several authors presented different numerical techniques of solving coupled system of second order boundary value problems [7-9].

In the next section we present the definition of cubic B-splines. In finite element method, the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. The finite element method viz., Galerkin method produces a weak form of approximate solution for a given differential equation and is unique under appropriate conditions [10,11] irrespective of properties of a given differential operator and weak solution is also a classical solution of given differential equation provided sufficient attention is given to the boundary conditions[12]. The attention to boundary conditions is presented in section 3. In section

Corresponding author's e-mail: kasi@nitw.ac.in

4 the Galerkin method to solve the given coupled linear system (1) has been presented. In section 5 the solution procedure to find the nodal parameters has been presented. In section 6 the proposed method is tested on two coupled linear systems and one coupled non linear system. The solution of coupled nonlinear system has been obtained as the limit of sequence of coupled linear problems generated by quasilinearization technique [4]. Finally, we presented the conclusions in the last section.

2 Definition of Cubic B-Splines

The existence of cubic B-spline interpolate $s(x)$ to a function $f(x)$ in a closed interval $[a,b]$ for spaced knots

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

is established by constructing it. The construction of $s(x)$ is done with the help of cubic B-splines. Introduce six additional knots x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3} such that

$$x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

Now the cubic B- splines $B_i(x)$, given in [13, 14] are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-2}^{i+2} \dfrac{(x_r - x)_+^3}{\pi'(x_r)} & \text{if } x \in [x_{i-2}, x_{i+2}] \\ 0 & \text{otherwise} \end{cases}$$

where

$$(x_r - x)_+^3 = \begin{cases} (x_r - x)^3 & \text{if } x_r \ge x \\ 0 & \text{if } x_r \le x \end{cases}$$

and

$$\pi(x) = (x - x_{i-2})(x - x_{i-1})(x - x_i)(x - x_{i+1})(x - x_{i+2}).$$

It can be shown that the set $\{B_{-1}(x), B_0(x), B_1(x), \ldots, B_n(x), B_{n+1}(x)\}$ forms a basis for the space $S_3(\pi)$ of cubic polynomial splines [15]. Schoenberg [16] has proved that the cubic B-splines are the unique non zero splines of smallest compact support with knots at

$$x_{-3} < x_{-2} < x_{-1} < x_0 < x_1 < \ldots < x_n < x_{n+1} < x_{n+2} < x_{n+3}.$$

3 Attention to Boundary Conditions

To solve the coupled linear system of boundary value problem (1) by Galerkin method with cubic B-splines as basis functions, we approximate the solutions u_1 and u_2 as

$$u_1(x) = \sum_{j=-1}^{n+1} \alpha_j B_j(x) \tag{3.1}$$

$$u_2(x) = \sum_{j=-1}^{n+1} \beta_j B_j(x) \tag{3.2}$$

where α_j's and β_j's are nodal parameters to be determined and $B_j(x)$'s are cubic B-spline basis functions.

When the chosen approximation satisfies the prescribed boundary conditions, it gives better approximation results. In view of this, the basis functions are redefined into a new set of basis functions which vanish on the boundary along with the nonhomogeneous part which takes care of the prescribed boundary conditions. In the set of cubic B-splines $\{B_{-1}(x), B_0(x), B_1(x), \ldots, B_{n-1}(x), B_n(x), B_{n+1}(x)\}$, the basis functions $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_{n-1}(x)$, $B_n(x)$ and $B_{n+1}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary. The procedure for redefining of the basis functions is as follows.

Applying the boundary conditions (1.3) for $u_1(x)$ and $u_2(x)$ defined in (2) and from the definition of cubic B-splines described in section 2, we get

$$\begin{aligned} u_{10} &= \sigma_1 u_1'(a) + \sigma_2 u_1(a) = \sigma_1 u_1'(x_0) + \sigma_2 u_1(x_0) \\ &= \sigma_1[\alpha_{-1} B_{-1}'(x_0) + \alpha_0 B_0'(x_0) + \alpha_1 B_1'(x_0)] \\ &\quad + \sigma_2[\alpha_{-1} B_{-1}(x_0) + \alpha_0 B_0(x_0) + \alpha_1 B_1(x_0)] \end{aligned} \tag{3.3}$$

$$\begin{aligned} u_{11} &= \sigma_3 u_1'(b) + \sigma_4 u_1(b) = \sigma_3 u_1'(x_n) + \sigma_4 u_1(x_n) \\ &= \sigma_3[\alpha_{n-1} B_{n-1}'(x_n) + \alpha_n B_n'(x_n) + \alpha_{n+1} B_{n+1}'(x_n)] \\ &\quad + \sigma_4[\alpha_{n-1} B_{n-1}(x_n) + \alpha_n B_n(x_n) + \alpha_{n+1} B_{n+1}(x_n)] \end{aligned} \tag{3.4}$$

$$\begin{aligned} u_{20} &= \mu_1 u_2'(a) + \mu_2 u_2(a) = \mu_1 u_2'(x_0) + \mu_2 u_2(x_0) \\ &= \mu_1[\beta_{-1} B_{-1}'(x_0) + \beta_0 B_0'(x_0) + \beta_1 B_1'(x_0)] \\ &\quad + \mu_2[\beta_{-1} B_{-1}(x_0) + \beta_0 B_0(x_0) + \beta_1 B_1(x_1)] \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_{21} &= \mu_3 u_2'(b) + \mu_4 u_2(b) = \mu_3 u_2'(x_n) + \mu_4 u_2(x_n) \\ &= \mu_3[\beta_{n-1} B_{n-1}'(x_n) + \beta_n B_n'(x_n) + \beta_{n+1} B_{n+1}'(x_n)] \\ &\quad + \mu_4[\beta_{n-1} B_{n-1}(x_n) + \beta_n B_n(x_n) + \beta_{n+1} B_{n+1}(x_n)] \end{aligned} \tag{3.6}$$

Now eliminating α_{-1}, α_{n+1}, β_{-1} and β_{n+1} from the equations (3.1) to (3.6), we get

$$u_1(x) = w_1(x) + \sum_{j=0}^{n} \alpha_j \tilde{B}_j(x) \tag{3.7}$$

$$u_2(x) = w_2(x) + \sum_{j=0}^{n} \beta_j \hat{B}_j(x) \tag{3.8}$$

where

$$\begin{aligned} w_1(x) &= \frac{u_{10}}{\sigma_1 B_{-1}'(x_0) + \sigma_2 B_{-1}(x_0)} B_{-1}(x) \\ &\quad + \frac{u_{11}}{\sigma_3 B_{n+1}'(x_n) + \sigma_4 B_{n+1}(x_n)} B_{n+1}(x) \end{aligned} \tag{3.9}$$

$$\begin{aligned} w_2(x) &= \frac{u_{20}}{\mu_1 B_{-1}'(x_0) + \mu_2 B_{-1}(x_0)} B_{-1}(x) \\ &\quad + \frac{u_{21}}{\mu_3 B_{n+1}'(x_n) + \mu_4 B_{n+1}(x_n)} B_{n+1}(x) \end{aligned} \tag{3.10}$$

$$\tilde{B}_j(x) = \begin{cases} B_j(x) - \dfrac{\sigma_1 B_j'(x_0) + \sigma_2 B_j(x_0)}{\sigma_1 B_{-1}'(x_0) + \sigma_2 B_{-1}(x_0)} B_{-1}(x) & \text{for } j = 0,1 \\ B_j(x) \text{ for } j = 2,3,\ldots,n-2 \\ B_j(x) - \dfrac{\sigma_3 B_j'(x_n) + \sigma_4 B_j(x_n)}{\sigma_3 B_{n+1}'(x_n) + \sigma_4 B_{n+1}(x_n)} B_{n+1}(x) & \text{for } j = n-1, n \end{cases} \tag{3.11}$$

$$\hat{B}_j(x)=\begin{cases} B_j(x)-\dfrac{\mu_1 B_j{}'(x_0)+\mu_2 B_j(x_0)}{\mu_1 B_{-1}{}'(x_0)+\mu_2 B_{-1}(x_0)}B_{-1}(x) & for \quad j=0,1 \\ B_j(x)\ for \quad j=2,3,...,n-2 \\ B_j(x)-\dfrac{\mu_3 B_j{}'(x_n)+\mu_4 B_j(x_n)}{\mu_3 B_{n+1}{}'(x_n)+\mu_4 B_{n+1}(x_n)}B_{n+1}(x) & for \quad j=n-1,n \end{cases} \tag{3.12}$$

The new set of basis functions for the approximations $u_1(x)$ and $u_2(x)$ are $\{\tilde{B}_j(x), j=0,1,...,n\}$ and $\{\hat{B}_j(x), j=0,1,...,n\}$ respectively. Here $w_1(x)$ and $w_2(x)$ take care of given set of boundary conditions and $\tilde{B}_j(x)$'s, $\hat{B}_j(x)$'s vanish on the boundary.

4 Description of the method

Applying the Galerkin method with the redefined set of basis functions $\tilde{B}_i(x)$ and $\hat{B}_i(x)$, $i = 0, 1,\ldots, n\text{-}1, n$ to the problem (1), we get

$$\int_{x_0}^{x_n}\left[a_0(x)u_1'' + a_1(x)u_1' + a_2(x)u_1\right]\tilde{B}_i(x)dx + \int_{x_0}^{x_n}\left[a_3(x)u_2'' + a_4(x)u_2' + a_5(x)u_2\right]\tilde{B}_i(x)dx = \int_{x_0}^{x_n} f_1(x)\tilde{B}_i(x)dx, \quad \text{for } i = 0, 1,\ldots, n \tag{4.1}$$

$$\int_{x_0}^{x_n}\left[b_0(x)u_1'' + b_1(x)u_1' + b_2(x)u_1\right]\hat{B}_i(x)dx + \int_{x_0}^{x_n}\left[b_3(x)u_2'' + b_4(x)u_2' + b_5(x)u_2\right]\hat{B}_i(x)dx = \int_{x_0}^{x_n} f_2(x)\hat{B}_i(x)dx, \quad \text{for } i = 0, 1,\ldots, n \tag{4.2}$$

Substitute the approximations for $u_1(x)$ and $u_2(x)$ given in (3.7) and (3.8) in (4.1) and (4.2), and after rearranging the terms for resulting equations, we get a coupled system of equations in the matrix form as

$$\mathbf{A}\,\alpha + \mathbf{B}\,\beta = \mathbf{F_1} \tag{4.3}$$

$$\mathbf{C}\,\alpha + \mathbf{D}\,\beta = \mathbf{F_2} \tag{4.4}$$

where

$$\mathbf{A} = [a_{ij}]; \tag{4.5}$$

$$a_{ij} = \int_{x_0}^{x_n}\{a_0(x)\frac{d^2}{dx^2}\left[\tilde{B}_j(x)\right] + a_1(x)\frac{d}{dx}\left[\tilde{B}_j(x)\right] + a_2(x)\tilde{B}_j(x)\}\tilde{B}_i(x)dx$$

for $i=0,1,2,\ldots,n$, $j=0,1,2,\ldots n$

$$\mathbf{B} = [b_{ij}]; \tag{4.6}$$

$$b_{ij} = \int_{x_0}^{x_n}\{a_3(x)\frac{d^2}{dx^2}\left[\hat{B}_j(x)\right] + a_4(x)\frac{d}{dx}\left[\hat{B}_j(x)\right] + a_5(x)\hat{B}_j(x)\}\tilde{B}_i(x)dx$$

for $i=0,1,2,\ldots,n$, $j=0,1,2,\ldots n$

$$\mathbf{C} = [c_{ij}]; \tag{4.7}$$

$$c_{ij} = \int_{x_0}^{x_n}\{b_0(x)\frac{d^2}{dx^2}\left[\tilde{B}_j(x)\right] + b_1(x)\frac{d}{dx}\left[\tilde{B}_j(x)\right] + b_2(x)\tilde{B}_j(x)\}\hat{B}_i(x)dx$$

for $i=0,1,2,\ldots,n$, $j=0,1,2,\ldots n$

$$\mathbf{D} = [d_{ij}]; \tag{4.8}$$

$$d_{ij} = \int_{x_0}^{x_n}\{b_3(x)\frac{d^2}{dx^2}\left[\hat{B}_j(x)\right] + b_4(x)\frac{d}{dx}\left[\hat{B}_j(x)\right] + b_5(x)\hat{B}_j(x)\}\hat{B}_i(x)dx$$

for $i=0,1,2,\ldots,n$, $j=0,1,2,\ldots n$

$$\mathbf{F_1} = [f_{1i}]; \tag{4.9}$$

$$f_{1i} = \int_{x_0}^{x_n}[f_1(x) - \{a_0(x)\frac{d^2w_1}{dx^2} + a_1(x)\frac{dw_1}{dx} + a_2(x)w_1(x) + a_3(x)\frac{d^2w_2}{dx^2} + a_4(x)\frac{dw_2}{dx} + a_5(x)w_2(x)\}]\tilde{B}_i(x)dx$$

$$\mathbf{F_2} = [f_{2i}]; \tag{4.10}$$

$$f_{2i} = \int_{x_0}^{x_n}[f_2(x) - \{b_0(x)\frac{d^2w_1}{dx^2} + b_1(x)\frac{dw_1}{dx} + b_2(x)w_1(x) + b_3(x)\frac{d^2w_2}{dx^2} + b_4(x)\frac{dw_2}{dx} + b_5(x)w_2(x)\}]\hat{B}_i(x)dx$$

and

$$\boldsymbol{\alpha} = [\alpha_0 \quad \alpha_1 \quad \cdots \quad \alpha_n]^T, \quad \boldsymbol{\beta} = [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_n]^T$$

5 Solution procedure to find the nodal parameters

A typical integral element in any of the matrices **A, B, C** and **D** is $\sum_{m=0}^{n-1} \mathrm{I}_m$,

where $\mathrm{I}_m = \int_{x_m}^{x_{m+1}} \mathrm{r}_i(x)\mathrm{r}_j(x)\mathrm{Z}(x)dx$ and $\mathrm{r}_i(x)$, $\mathrm{r}_j(x)$ are the basis functions or their derivatives. It may be noted that $\mathrm{I}_m = 0$ if $(x_{i-2}, x_{i+2}) \cap (x_{j-2}, x_{j+2}) \cap (x_m, x_{m+1}) = \phi$. To evaluate each I_m, we employed 4-point Gauss-Legendre quadrature formula. Thus the matrices **A, B, C** and **D** described in (4.5) to (4.8) are seven diagonal band matrices. We solve the coupled system of equations (4.3)-(4.4) by using the following iteration formula.

$$\mathbf{A}\,\alpha^{(r+1)} = \mathbf{F_1} - \mathbf{B}\,\beta^{(r)}$$

$$\mathbf{D}\,\beta^{(r+1)} = \mathbf{F_2} - \mathbf{C}\,\alpha^{(r+1)}, \quad r = 0,1,2,\ldots$$

The nodal parameters α_i's and β_i's can be obtained from the above system by using the band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1) by the proposed method.

6 Numerical Examples

To test the applicability of the proposed method, we considered two examples of coupled linear systems and one coupled non linear system. The numerical results for these examples are compared with the results available in the literature.

Example 1:
Consider the following coupled linear system of boundary value problem

$$u''(x) + u'(x) + xu(x) + v'(x) + 2xv(x) = f_1(x), \quad 0 \le x \le 1, \tag{5.1}$$

$$2u'(x) + x^2 u(x) + v''(x) + v(x) = f_2(x), \quad 0 \le x \le 1, \tag{5.2}$$

subject to $u(0) = u(1) = 0$, $v(0) = v(1) = 0$ (5.3)
where

$$f_1(x) = -2(1+x)\cos x + \pi\cos\pi x + 2x\sin\pi x + (4x - 2x^2 - 4)\sin x$$

$$f_2(x) = 4(1-x)\cos x + 2(-2 + x^2 - x^3)\sin x + (1-\pi^2)\sin\pi x$$

The exact solutions of u and v are $u(x) = 2(1 - x)\sin(x)$ and $v(x) = \sin(\pi x)$. The proposed method is tested on the problem (5.1) to (5.3). Numerical results obtained by the proposed method are presented in table1 and table 2 and compared with the results in [5] and [6]. The maximum absolute error obtained by the proposed method is compared with that of obtained in [5] and [6] in table 3.

Table 1
Numerical results for the variable u in Example 1

x	Absolute error by proposed method with 10 intervals	Absolute error by the method [5] with 32 intervals	Absolute error by the method [6] with 32 intervals
0.08	8.18×10^{-6}	3.3×10^{-3}	3.2×10^{-3}
0.24	1.65×10^{-5}	7.7×10^{-3}	9.2×10^{-4}
0.40	1.66×10^{-5}	9.7×10^{-3}	2.0×10^{-3}
0.56	1.20×10^{-5}	9.5×10^{-3}	2.2×10^{-4}
0.72	5.71×10^{-5}	7.3×10^{-3}	4.1×10^{-3}
0.88	9.65×10^{-6}	3.4×10^{-3}	1.0×10^{-2}
0.96	1.41×10^{-6}	1.1×10^{-3}	1.1×10^{-3}

Table 2
Numerical results for the variable v in Example 1

x	Absolute error by proposed method with 10 intervals	Absolute error by the method [5] with 32 intervals	Absolute error by the method [6] with 32 intervals
0.08	2.32×10^{-5}	7.7×10^{-3}	1.5×10^{-3}
0.24	6.22×10^{-5}	2.0×10^{-2}	7.0×10^{-3}
0.40	8.38×10^{-5}	2.7×10^{-2}	7.4×10^{-3}
0.56	8.52×10^{-5}	2.7×10^{-2}	1.0×10^{-2}
0.72	6.60×10^{-5}	2.0×10^{-2}	4.4×10^{-3}
0.88	3.14×10^{-5}	9.4×10^{-3}	2.1×10^{-2}
0.96	1.07×10^{-5}	3.1×10^{-3}	6.9×10^{-3}

Table 3
Maximum Absolute Errors for the Example 1

Variable	By the proposed method with 10 intervals	By the method in [5] with 32 intervals	By the method in [6] with 32 intervals
u	1.66×10^{-5}	9.7×10^{-3}	1.0×10^{-2}
v	8.52×10^{-5}	2.7×10^{-2}	2.1×10^{-2}

Example 2:
Consider the coupled linear system of boundary value problem

$$u'' + xu + 2v' = x^3 + x^2 + 2 + 2\cos x, \quad 0 \le x \le 1, \tag{5.4}$$
$$u + v'' + 2v = x^2 + x + \sin x$$

subject to the boundary conditions

$$u'(0) + u(0) = 1$$
$$v'(1) + v(1) = \cos 1 + \sin 1 \tag{5.5}$$
$$u(1) = 2, v'(0) = 1$$

The exact solutions u and v for the system (5.4) and (5.5) are $u(x) = x^2 + x$ and $v(x) = \sin x$. The proposed method is tested on the problem (5.1) to (5.3) where the domain [0,1] is divided into 10 equal intervals. Numerical results obtained by the proposed method are shown along with the exact solutions are presented in table 4.

Table 4

x	Absolute error for $u(x)$	Absolute error for $v(x)$
0.0	0.37784E-05	0.17698E-05
0.1	0.34035E-05	0.17363E-05
0.2	0.30409E-05	0.16417E-05
0.3	0.26974E-05	0.14988E-05
0.4	0.23735E-05	0.13226E-05
0.5	0.20593E-05	0.11247E-05
0.6	0.17440E-05	0.92328E-06
0.7	0.14138E-05	0.73612E-06
0.8	0.10335E-05	0.57280E-06
0.9	0.57340E-06	0.44882E-06
1.0	0.00000E+00	0.37789E-06

Example 3:
Consider the following coupled nonlinear system of boundary value problem

$$u''(x) + xu(x) + 2xv(x) + xu^2(x) = f_1(x)\,,\quad 0 \le x \le 1 \qquad (5.6)$$

$$x^2u(x) + v'(x) + v(x) + \sin(x)v^2(x) = f_2(x)\,,\quad 0 \le x \le 1 \qquad (5.7)$$

subject to $u(0) = u(1) = 0,\ v(0) = v(1) = 0$ (5.8)

where $f_1(x) = 2x\sin\pi x + x^5 - 2x^4 + x^2 - 2$

and

$$f_2(x) = x^3(1-x) + \sin\pi x(1 + \sin x \sin\pi x) + \pi\cos\pi x$$

.

The exact solutions u and v for the (5.6)-(5.8) are $u(x) = x - x^2$ and $v(x) = \sin(\pi x)$. By the quasilinearization technique [4] the problem (5.6)-(5.8) has been converted into a sequence of coupled linear problems as

$$u''_{n+1}(x) + \left[x(1+2u_n)\right]u_{n+1}(x) + 2xv_{n+1}(x) = f_1(x) + xu_n^2(x) \qquad (5.9)$$

$$x^2u_{n+1}(x) + v'_{n+1}(x) + \left[1 + 2\sin(x)v_n(x)\right]v_{n+1}(x) = f_2(x) + \sin(x)v_n^2(x)\,, \qquad (5.10)$$

subject to $u_{n+1}(0) = u_{n+1}(1) = 0,\ v_{n+1}(0) = v_{n+1}(1) = 0$ for $n = 0,1,2,\ldots$

Here u_{n+1} and v_{n+1} are the $(n+1)^{\text{th}}$ approximations for u and v respectively. The proposed method is tested for the above problem. Numerical results obtained by the proposed method are presented in table 5 and table 6 and compared with the results in [5] and [6]. The maximum absolute error obtained by the proposed method is compared with that of obtained in [5] and [6] in table 7.

Table 5
Numerical results for the variable u in Example 3

x	Absolute error by proposed method with 10 intervals	Absolute error by the method [5] with 32 intervals	Absolute error by the method [6] with 32 intervals
0.08	3.27×10^{-6}	5.0×10^{-4}	1.4×10^{-4}
0.24	9.40×10^{-6}	1.4×10^{-3}	4.4×10^{-5}
0.40	1.46×10^{-5}	2.1×10^{-3}	6.7×10^{-5}
0.56	1.84×10^{-5}	2.2×10^{-3}	9.3×10^{-5}
0.72	1.89×10^{-5}	1.8×10^{-3}	4.9×10^{-5}
0.88	1.27×10^{-5}	9.0×10^{-4}	8.6×10^{-5}
0.96	4.99×10^{-6}	3.0×10^{-4}	7.1×10^{-5}

Table 6
Numerical results for the variable v in Example 3

x	Absolute error by proposed method with 10 intervals	Absolute error by the method [5] with 32 intervals	Absolute error by the method [6] with 32 intervals
0.08	1.64×10^{-5}	2.0×10^{-3}	2.4×10^{-4}
0.24	7.63×10^{-6}	5.6×10^{-3}	2.3×10^{-3}
0.40	4.90×10^{-5}	7.9×10^{-3}	8.9×10^{-4}
0.56	7.39×10^{-6}	8.2×10^{-3}	1.4×10^{-3}
0.72	1.76×10^{-5}	6.5×10^{-3}	3.1×10^{-3}
0.88	4.16×10^{-5}	3.1×10^{-3}	1.6×10^{-3}
0.96	1.79×10^{-5}	1.0×10^{-3}	9.8×10^{-4}

Table 7
Maximum Absolute Errors for the Example 3

Variable	By the proposed Method with 10 intervals	By the method in [5] with 32 intervals	By the method in [6] with 32 intervals
u	1.89×10^{-5}	2.2×10^{-3}	1.4×10^{-4}
v	4.16×10^{-5}	8.2×10^{-3}	3.1×10^{-3}

7 Conclusions

In this paper, we have developed a Galerkin method with cubic B-splines as basis functions to solve a coupled system of linear second order boundary value problems. The cubic B-splines basis set have been redefined into a new set of basis functions which vanish on the boundary along with the nonhomogeneous part which takes care of the prescribed boundary conditions. The solution of a coupled nonlinear system has been obtained as the limit of sequence of coupled linear problems generated by quasilinearization technique. The proposed method is applied to solve two linear problems and one nonlinear problem to test the efficiency of the method. The numerical results obtained by the method are in good agreement with the exact solutions available in the literature. The maximum absolute errors obtained by the proposed method are less when compared with those of available in the literature. The objective of this paper is to present a simple technique to solve a coupled system of second order boundary value problems.

References

[1]. C Wafo soh, F.M. Mahomed, Internat. J. Non-Linear Mech. **36**, 671-677 (2001)
[2]. S Matthews, Parameter robust numerical methods for a system of two coupled singularly perturbed reaction diffusion equations, *Master Thesis*, School of Mathematical Sciences, Dublin city University, (2000)
[3]. K.N. Tomantschger, J. comput. Appl. Math. **140**, 773-783 (2002)

[4]. R.E Bellman and R.E. Kalaba, *Quasilinearization and Nonlinear Boundary value problems,* American Elsevier, New York (1965).
[5]. Fazhan Geng, Minggen Cui, J. Math. Anal. Appl. **327**, 1167-1181 (2007)
[6]. Mehdi Dehgan, Abbas Saadatmandi, Math. Comput. Modelling **46**, 1434-1441 (2007)
[7]. A. Tamilselvan, N. Ramanujan, V. Shanti, J. comput. Appl. Math. **202**, 203-216 (2007)
[8]. H. B. Thompson, C. Tisdell, J. Math. Anal. Appl. **248**, 333-347, (2000)
[9]. T. Valanarasu, N. Ramanujan, Appl. Math. Comput. **147(6)**, 227-240, (2004)
[10]. Lions J.L. and Magenes E, *Non-Homogeneous Boundary Value Problems and Applications,* Springer – Verlag, Berlin (1972).
[11]. Bers L.,John F. and Schecheter M., *Partial Differential Equations,* John Wiley Inter Science, New York, 1964.
[12]. Mitchell A.R. and Wait R., *The Finite Element Method in Partial Differential Equations*, John Wiley and Sons, London (1977).
[13]. M.G. Cox, Jour. Inst. Mathematics and Appl. **10**, 134-149, (1972)
[14]. Carl de Boor, *A practical guide to splines,* Springer-Verlag (1978).
[15]. P. M. Prenter, *Splines and Variational Methods,* John-Wiley and Sons, New York, (1975)
[16]. I.J.Schoenberg, *On Spline Functions,* MRC Report 625, University of Wisconsin (1966).