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Features of mathematical modelling in the analysis of console-type structures

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Abstract. The article deals with the mathematical model of console-type structural elements. The dynamic load is presented as quasi-static one. The differential equation of bending of an object is nonlinear and has movable singular points in which the solution has discontinuity. From a physical point of view, the object will break (collapse) in this place. The application of the majorant method to the solution of the problem allows, in contrast to the classical approach, establishing the boundaries of the solution area and to construct an analytical approximate solution to the problem with a given accuracy. As a result, it's possible to calculate the displacement at any point of the cantilever structure and estimate the stress-strain state of the object.

1. Introduction

Mathematical models are actively used in different options and in various fields for the study of technical 5-24, economic 25, 26, biological 27-29 processes and medicine 30, 31, as a rigorous and accurate method for substantiating the results obtained.

In this paper, on the basis of mathematical modeling, the problem of bending the console with an instantaneous load application is presented. Dynamic load is considered as quasi-static one. High-rise construction under certain assumptions and, for example, with the presence of "building core", can also be considered as an object of study. The mathematical model is a nonlinear differential equation

$$y'' = \frac{M_x}{EJ_x}\sqrt{(1+(y')^2)^3} + F_x, \tag{1}$$

here M_x – bending moment; EJ_x – stiffness uniform beam; F_x – external action. The presented mathematical model (1) can reflect in a quasi-static form the process of influence on the object of various external loads: a sudden shock wave, contact interaction with a moving object, etc.

Equation (1) in a general way is unsolvable in quadratures. It is possible to note the works when such nonlinear differential equations in particular cases are solvable in quadratures 6-8]. Such equations have movable singular points, in which the solution is discontinuous. From the physical point of view, in the first case we get a break of the

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console, in the second – the failure of the object. In both cases, the movable singular points indicate the site of the object's failure.

2. The mathematical treatment of problem for the nonlinear differential equation

Since mathematical models represent differential equations, we characterize them from a mathematical point of view. The area of existence of the solution of equation (1) is divided into the area of analyticity and vicinity of movable singular points, the finding of which requires the development of special theory and calculation technology. At the first stage for equation (1), it is necessary to solve the problem of existence and uniqueness of the solution in the field of analyticity, which would allow obtaining an analytical approximate solution with a given accuracy. It should be noted that the existing classical Cauchy theorem does not work in this case.

In the present paper, for the considered class of nonlinear differential equations in a complex domain the existential theorem in holomorphic region has been proved, analytical approximate solution has been designed as well as the influence of perturbation of initial conditions on the latter has been researched. This problem arises when we apply analytic continuation of the solution. It should be pointed out that the existing classical Cauchy's theorem doesn't provide means for obtaining the presented results.

The structure of equation (1) allows simplifying it without loss of the solution of the initial variant. Denote

$$y' = Y , \qquad M_x / (EJ_x) = \psi(x)$$

then equation (1) takes the form of

$$Y' = \psi(x)\sqrt{(1+Y^2)^3} + F(x) \tag{2}$$

Add to it is initial data

$$Y(x_0) = Y_0 \tag{3}$$

We formulate and prove a theorem of existence and uniqueness of the solution of the Cauchy problem (2)-(3). In the proof of the theorem we use the proprietary technology [14].

2.1 Theorem

Let us assume that:

1) functions $\psi(x)$ and $F(x) \in C^{\infty}$ in range $|x - x_0| < \rho_1$, $\rho_1 = \text{const} \neq 0$;

2) exist $M_{1,n}$ and $M_{2,n}$, with

$$\left|\frac{\psi^{(n)}(x_0)}{n!}\right| \le M_{1,n} \qquad \left|\frac{F^{(n)}(x_0)}{n!}\right| \le M_{2,n} \qquad \forall\, n = 0, 1, 2, \ldots$$

where is $M_{1,n}$ and $M_{2,n}$ are some constants. Then there is a unique solution of the problem (2)-(3) in the form

$$Y(x)=\sum_{0}^{\infty}C_n(x-x_0)^n \tag{4}$$

in the area of $|x-x_0|<\rho_2$, where

$$\rho_2=\min\left\{\rho_1,\frac{1}{3(M+1)^3}\right\} \qquad M=\max\{|Y(x_0)|,M_{1,n},M_{2,n}\} \qquad n=0,1,2$$

2.2. The proof

According to the theorem we have

$$\psi(x)=\sum_{0}^{\infty}D_n(x-x_0)^n \qquad F(x)=\sum_{0}^{\infty}A_n(x-x_0)^n \tag{5}$$

Introduce the notation:

$$Y^2=\sum_{0}^{\infty}C_n^{*}(x-x_0)^n \qquad 1+Y^2=\sum_{0}^{\infty}C_{1,n}^{**}(x-x_0)^n$$

$$C_n^{*}=\sum_{0}^{n}C_{n-i}C_i \qquad C_{1,0}^{**}=C_0^{*}+1 \qquad \forall\, n=1,2,\ldots \qquad C_{1,n}^{**}=C_n^{*} \qquad \sqrt{(1+Y^2)^3}=\sum_{0}^{\infty}B_n(x-x_0)^n$$

$$(1+Y^2)^3=\sum_{0}^{\infty}B_n^{*}(x-x_0)^n \qquad B_n^{*}=\sum_{0}^{\infty}B_{n-i}B_i$$

Substitute (4) and (5) in equation (2), we obtain

$$\sum_{1}^{\infty}C_n n(x-x_0)^{n-1}=\sqrt{\left(1+\left(\sum_{0}^{\infty}C_n(x-x_0)^n\right)^2\right)^3}\sum_{0}^{\infty}D_n(x-x_0)^n+\sum_{0}^{\infty}A_n(x-x_0)^n \tag{6}$$

From (6), taking into account the introduced designations for the coefficients B_n and C_n, we obtain recurrence relations:

$$B_n=\frac{1}{2B_0}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n-i}C_{1,n-j-i}^{**}C_{1,j}^{**}\right)C_{1,i}^{**}-\sum_{i=1}^{n-1}B_{n-i}B_i\right) \tag{7}$$

$$C_n=\frac{1}{n}\left(\sum_{i=0}^{n-1}D_iB_{n-1-i}+A_{n-1}\right) \tag{8}$$

wherein $B_0=\sqrt{(C_0^2+1)^3}$. Recurrence relations (7) and (8) allow unambiguously to obtain expressions for coefficients C_n:

$$C_1=D_0\sqrt{(C_0^2+1)^3}+A_0 \qquad B_1=3C_0C_1\sqrt{C_0^2+1}$$

$$C_2 = \frac{\sqrt{C_0^2+1}}{2}(3D_0C_0C_1 + D_1(C_0^2+1) + A_1)$$

$$B_2 = \frac{3C_0}{\sqrt{C_0^2+1}}(C_2C_0^2 + C_2 + C_0C_1^2)$$

$$C_3 = \frac{1}{3}\left(D_0 \frac{3C_0}{\sqrt{(C_0^2+1)^3}}(C_2C_0^2 + C_2 + C_0C_1^2) + D_1 \cdot 3C_0C_1\sqrt{C_0^2+1} + D_2\sqrt{(C_0^2+1)^3} \right) \text{ etc.}$$

We prove for the coefficients B_n and C_n of the estimate

$$|B_n| \le 3^{n+1}(M+1)^{3(n+1)} \tag{9}$$

$$|C_n| \le 3^n M(M+1)^{3n} \tag{10}$$

In the proofs, we use a number of notations due to the complexity of the structure of the original equation (2):

$$Y^2 = \sum_0^\infty C_n^*(x-x_0)^n \quad 1+Y^2 = \sum_0^\infty C_{1,n}^{**}(x-x_0)^n$$

$$C_n^* = \sum_0^n C_{n-i}C_i \quad C_{1,0}^{**} = C_0^* + 1 \quad C_{1,n}^{**} = C_n^* \ \forall\, n = 1, 2, ...$$

$$(1+Y^2)^3 = \sum_0^\infty C_{3,n}^{**}(x-x_0)^n \quad C_{2,n}^{**} = \sum_{i=0}^n C_{1,n-i}^{**}C_{1,i}^{**}$$

$$C_{3,n}^{**} = \sum_{i=0}^n C_{2,n-i}^{**}C_{1,i}^{**} = \sum_{i=0}^n C_{2,n-i}^{**}C_{1,i}^{**} = \sum_{i=0}^n \left(\sum_{j=0}^{n-i} C_{1,n-i-j}^{**}C_{1,j}^{**} \right) C_{1,i}^{**}$$

$$\sqrt{(1+Y^2)^3} = \sum_0^\infty B_n(x-x_0)^n \quad (1+Y^2)^3 = \sum_0^\infty B_n^*(x-x_0)^n$$

$$B_n^* = \sum_{i=0}^n B_{n-i}B_i$$

Based on equation (2) and the introduced notations for B_0, we obtain

$$B_0 = \sqrt{(C_0^2+1)^3}$$

The recurrence relations (7) and (8) determine the algorithm for finding the coefficients and B_n and C_n Firstly, the coefficient B_n is determined, and then the coefficient C_{n+1} for $n = 0, 1, 2, ...$ The calculations are presented with intermediate results to observe the transitions in the notations. We prove the validity of the estimate (9) for B_n. From the ratio (7) we obtain:

$$\left|B_n\right| \le \left|\frac{1}{2B_0}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n-i} C_{1,n-i-j}^{**} C_{1,j}^{**}\right) C_{1,i}^{**} - \sum_{i=1}^{n-1} B_{n-i} B_i\right)\right| \le \left|\frac{1}{2B_0}\left(\sum_{i=0}^{n}\left(\sum_{j=0}^{n-i} C_{1,n-i-j}^{**} C_{1,j}^{**}\right) C_{1,i}^{**}\right)\right| \le$$

$$\le \left|\frac{1}{2B_0}\left(\sum_{j=0}^{n} C_{1,n-j}^{**} C_{1,j}^{**} C_{1,0}^{**} + (C_{1,0}^{**})^2 C_{1,n}^{**} + \sum_{i=0}^{n-1}\left(\sum_{j=0}^{n-i-1} C_{1,n-1-i-j}^{**} C_{1,j}^{**}\right) C_{1,i}^{**}\right)\right| \le$$

$$\le \left|\frac{1}{2B_0}\left(3(C_{1,0}^{**})^2 C_{1,n}^{**} + C_{1,0}^{**} \sum_{j=1}^{n-1} C_{1,n-j}^{**} C_{1,j}^{**} + 2(C_{1,0}^{**})^2 C_{1,n-1}^{**} + \right.\right.$$

$$\left.\left. + 2\sum_{i=1}^{n-2} C_{1,0}^{**} C_{1,n-i-1}^{**} C_{1,i}^{**} + \sum_{i=1}^{n-1} C_{1,i}^{**}\left(\sum_{j=1}^{n-i-2} C_{1,n-i-1-j}^{**} C_{1,j}^{**}\right)\right)\right| \le$$

$$\le \left|\frac{1}{2B_0}\left(3(C_0^* + 1)C_n^* + (C_0^* + 1)\sum_{j=1}^{n-1} C_{n-j}^* C_j^* + 2(C_0^* + 1)^2 C_{n-1}^* + \right.\right.$$

$$\left.\left. + 2\sum_{i=1}^{n-2} C_0^* C_{n-i-1}^* C_i^* + \sum_{i=1}^{n-1} C_i^*\left(\sum_{j=1}^{n-i-2} C_{n-i-j-1}^* C_j^*\right)\right)\right| \le$$

$$\le \left|\frac{1}{2(C_0^2+1)^{3/2}}\left(3(C_0^2+1)\sum_{i=0}^{n} C_{n-i} C_i + (C_0^2+1)\sum_{j=1}^{n-1}\left(\sum_{k=0}^{n-j-1} C_{n-j-1-k} C_k\right)\left(\sum_{k=0}^{j} C_{j-k} C_k\right) + \right.\right.$$

$$+ 2(C_0^2+1)\sum_{k=0}^{n-1} C_{n-1-k} C_k + 2\sum_{i=1}^{n-2} C_0^2\left(\sum_{k=0}^{n-i-1} C_{n-i-1-k} C_k\right)\left(\sum_{k=0}^{i} C_{i-k} C_k\right) +$$

$$\left.\left. + \sum_{i=1}^{n-1}\left(\sum_{k=0}^{i} C_{i-k} C_k\right)\left(\sum_{j=1}^{n-i-2}\left(\sum_{k=0}^{n-i-j-1} C_{n-i-j-1-k} C_k\right)\left(\sum_{k=0}^{j} C_{j-k} C_k\right)\right)\right)\right|$$

By performing the corresponding transformations in the last expression and substituting the expected estimates for the coefficients C_n in it, we get

$$\left|B_n\right| \le \frac{3}{2}\cdot 3^n M(M+1)^{3n}(n+1) + \frac{1}{2}\cdot 3^{n-1} M^3 (M+1)^{3(n-1)}(n-1)\frac{n}{2} +$$

$$+ n\cdot 3^{n-1}(M^2+1)^{1/2} M^2 (M+1)^{3(n-1)} + 3^{n-1} M(M+1)^{3(n-1)}\frac{(n-2)^2}{2} +$$

$$+ 3^{n-1} M(M+1)^{3(n-1)}(n-1)^2(n-3)^2 \le 3^{n+1}(M+1)^{3(n+1)}$$

Wherein $M = \max\{\left|Y(x_0)\right|, M_{1,n}, M_{2,n}\}$, $n = 0, 1, 2, \ldots$. Further from (8) follows

$$\left|C_n\right| \le \frac{1}{n}\left|\sum_{i=0}^{n-1} D_i B_{n-i-1} + A_{n-1}\right|$$

or, taking into account the estimates for the coefficients A_n, B_n, D_n,

$$\left|C_n\right| \le \frac{1}{n}\left(\sum_{i=0}^{n-1} M \cdot 3^{n-i}(M+1)^{3(n-i)} + M\right) \le M \cdot 3^n (M+1)^{3n}$$

we obtain a proof of the estimates (10).

Consider a series

$$\sum_{0}^{\infty} M \cdot 3^n (M+1)^{3n} (x-x_0)^n \tag{11}$$

which is majorizing for series (4). On the basis of a sufficient sign of convergence for the series (11), we obtain the region

$$\left|x-x_0\right| < \frac{1}{3(M+1)^3}$$

which is also the convergence region of the series (4).

Thus, the existence and uniqueness theorem in the field of analyticity is proved.

3. Conclusion

Solved by the authors of the problem allows you to create mathematical models of complex structures and phenomena. The obtained results allow to carry out the analytical continuation of the approximate solution with a given accuracy. In which case, a posteriori error allows obtaining significantly more accurate a priori error.

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