

The stochastic finite element method in the assessment of bridge infrastructure objects - review

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Abstract. The stochastic finite element method (SFEM) is an extension of classical FEM which allows the representation of various types and sources of uncertainty in one computational system. This review paper presents information regarding SFEM implementation in the assessment of bridge objects. A concise theoretical background of the three most commonly used branches of SFEM is also presented. This technique is used in the assessment of bridge structures with regards to, e.g. load in motion problems, wind and seismic excitation analysis, random material property analysis, reliability and fatigue reliability analysis. However, it seems that the main feasibility concern is the lack of proper SFEM implementation into general purpose FEM systems which are used by bridge engineers. This is why, arguably, one of the most effective ways to introduce SFEM in real-life bridge engineering problems is with the methods that rely on multiple calculations of classical deterministic FEM e.g. Monte Carlo Simulation, or Response Function Method. To introduce randomness directly into individual finite elements of the model in a complete SFEM procedure it is usually necessary to develop proprietary computer programs.

1 Introduction

Bridge objects are key components of road infrastructure systems. Any serviceability failures of these structures often result in extremely costly renovation programs and troublesome hindrances to traffic flow. These consequences can seriously affect the transportation system efficiency of major cities especially when a failure concerns bridges in the city bypass or other fundamental communication routes.

To minimize the probability of the occurrence of these problems, special attention should be paid to the design stage of new bridges and thorough the assessment of existing ones prior to, e.g. renovation to increase their load capacity. Design costs are a very small fraction of the overall structures life cycle costs [1]. Thus, it is worthwhile investing time and resources for deep and reliable design or assessment analyses. For that purpose, in recent decades, probabilistic tools are more and more confidently implemented. Thanks to this, it is possible to properly represent important uncertainties in the analysis such as: random material

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properties, random loads and dynamic excitations, random executive imperfections and others.

An important branch of these methods is the Stochastic Finite Element Method (SFEM) [2, 3] which is an extension of its deterministic ancestor FEM. Depending on the technique of uncertainty propagation representation, it is developed in several variants, mainly: (I) FEM with Stochastic Perturbation Method (hereinafter referred to as Perturbation SFEM) [4], (II) Spectral Stochastic Finite Element Method (SSFEM) [5], (III) FEM with Monte Carlo Simulation (MCS) Technique often with a supplement of Latin Hypercube Sampling (LHS) or other sampling techniques [6]. These methods have recently been implemented in the analysis of various types of bridge superstructures and in the aspect of different types of analysis (modal, fatigue, static, reliability, etc.)

Therefore, the main objective of this paper is to collect and analyse information regarding: (I) a concise outline of theoretical background of individual SFEM techniques, (II) a review of analyses and their types in which SFEM was effectively implemented for the assessment of bridge structures response, (III) the identification of the aspects in which SFEM requires further research and development or is currently infeasible in real-life engineering problems.

2 Theoretical background

2.1 Random finite element method with perturbation approach

In the perturbation method, Taylor series expansion is used to formulate relationships between chosen characteristics of a random response and random structural parameters. Instability of the approximate solution appears in higher order terms [5]. One of the first applications of the perturbation approach for engineering problems can be found, for example, in [7]. In [8] and [9] this approach was further developed to be compatible with FEM. The second-order version of the stochastic finite element method was practically implemented in the aspect of structural analysis by Kleiber and Hien [4].

Using this approach to, for example, introduce random material properties into the system will result in the following form of the stiffness matrix [2]:

$$K = K_0 + \sum_{i=1}^N K_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N K_{ij}^{II} \alpha_i \alpha_j + \dots \quad (1)$$

where: α_i ($i = 1, 2, \dots, N$) are zero mean random variables representing system uncertainties after random field discretisation; and K_0 is a matrix containing expected values. K_i^I and K_{ij}^{II} can be derived as follows:

$$K_i^I = \left. \frac{\partial K}{\partial \alpha_i} \right|_{\alpha=0} \quad (2)$$

$$K_{ij}^{II} = \left. \frac{\partial^2 K}{\partial \alpha_i \partial \alpha_j} \right|_{\alpha=0} \quad (3)$$

Analogical considerations can be made in the case of the uncertainty in loads:

$$P = P_0 + \sum_{i=1}^N P_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N P_{ij}^{II} \alpha_i \alpha_j + \dots \quad (4)$$

and displacements:

$$U = U_0 + \sum_{i=1}^N U_i^I \alpha_i + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N U_{ij}^{II} \alpha_i \alpha_j + \dots \quad (5)$$

The displacement vector U can also be calculated using an iterative scheme [3]:

$$U_0 = K_0^{-1}P_0, \tag{6a}$$

$$U_i^I = K_0^{-1}(P_i^I - K_i^I U_0), \tag{6b}$$

$$U_{ij}^{II} = K_0^{-1}(P_{ij}^{II} - K_i^I U_i^I - K_j^I U_j^I - K_{ij}^{II} U_0), \tag{6c}$$

where: U_0 is equal to deterministic nodal displacements, U_i^I, U_{ij}^{II} are the first and second-order perturbation of the displacement vector, respectively.

In the case of the assumption that random variables α_i, α_j are Gaussian, the approximation of the response variability can be adopted as follows:

$$E(U) \approx U_0 + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N U_{ij}^{II} Cov[\alpha_i, \alpha_j] \tag{7}$$

$$Cov[U, U] \approx \sum_{i=1}^N \sum_{j=1}^N U_i^I (U_j^I)^T Cov[\alpha_i, \alpha_j] \tag{8}$$

where: $E(U)$ and $Cov[U, U]$ are the approximations of the expected values and covariance matrices of the random response (displacement), respectively.

The main drawback of second-order calculations is that it can only reliably represent phenomena with relatively low variability (coefficient of variation < 15%). In other cases higher order approximations must be used. This led to development of the recursive stochastic finite element method [10] and the generalized perturbation method with the response function method which are presented in [11]. The latter was done with the use of the advanced symbolic Maple® system, and was supported by several numerical examples.

2.2 Spectral Stochastic Finite Element Method (SSFEM)

The Spectral Stochastic Finite Element Method (SSFEM) was introduced by Ghanem and Spanos [5]. SSFEM is a development of the deterministic FEM to accommodate random functions. One of the main goals of this method is to represent random material properties in a more computationally efficient manner than Monte-Carlo simulation. In [5] a random, special variation of material properties in a structure as described by a Gaussian stochastic field which is generated using Karhunen-Loève (K-L) expansion (presented in e.g. [3], initially derived in [12, 13, 14]):

$$f(x, \theta) = \bar{f}(x) + \sum_{i=1}^{\infty} [\xi_i(\theta) \sqrt{\lambda_i} \psi_i(x)] \tag{9}$$

where: $\bar{f}(x)$ is the mean of random field/process, $\xi_i(\theta)$ is a group of uncorrelated random variables, λ_i and $\psi_i(x)$ are eigenvalues and eigenfunctions of the covariance kernel, respectively.

On this basis, local (element) stiffness matrices in SSFEM are in the following form [3]:

$$K^e(\theta) = K_0^e + \sum_{i=1}^{\infty} K_i^e \xi_i(\theta) \tag{10}$$

where: K_0^e is mean value of $K^e(\theta)$ and K_i^e is given by:

$$K_i^e = \sqrt{\lambda_i} \int_{\Omega_e} \psi_i(x) B^T D_0 B d\Omega_e \tag{11}$$

where: D_0 is the mean value of the constitutive matrix, while B is the strain–displacement matrix.

After aggregation of local stiffness matrices into a global form (using rectangular permutation matrices known from classical FEM), and with the assumption that loading is deterministic, the equilibrium equation is as follows:

$$[K_0 + \sum_{i=1}^{\infty} K_i \xi_i(\theta)]U(\theta) = F \quad (12)$$

Alas, the covariance function of solution $U(\theta)$ (which is the random vector of nodal displacements) and corresponding eigenfunctions are unknown. Therefore, unlike in perturbation FEM, SSFEM requires using a second type of extension to express the response process (e.g. displacement). In [5] improved Neumann Expansion or/and Homogenous Chaos Expansions are used and compared. In the case of the latter one, the appropriate expression is as follows:

$$U(\theta) = a_0 \Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1} \Gamma_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{i_1} a_{i_1 i_2} \Gamma_2(\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \dots \quad (13)$$

where: $\Gamma_n(\xi_{i_1}(\theta), \dots, \xi_{i_n}(\theta))$ is the n -th order polynomial chaos of the variables $(\xi_{i_1}(\theta), \dots, \xi_{i_n}(\theta))$. The above is often re-written into an alternative form by limiting series to p terms and mapping functional $\Gamma_n(\cdot)$ into $\Psi_j(\cdot)$ and coefficients a_i into U_j in one-to-one manner:

$$U(\theta) = \sum_{j=0}^p U_j \Psi_j(\theta) \quad (14)$$

On the basis of eq. 10 to eq. 14, the final equilibrium equation is constructed and solved as a linear system of algebraic equations, which, in turn, is obtained by a search of the optimal approximation of the exact solution $U(\theta)$ [3]. For further details on constructing polynomial chaos and mapping eq. 13 to eq. 14, the reader is advised to refer to the original text book [5] where several examples are presented.

2.3 Monte Carlo Simulation

The third, most straightforward approach to SFEM is a merger of the Monte Carlo Simulation (MCS) with deterministic FEM. It is done with the following steps [2, 15]:

- adoption / recognition of sources of uncertainty,
- selection of random variables, analysis of their distributions and correlation,
- adequate generation of a set of random fields,
- construction of NS deterministic FEM model variants in accordance with the generated fields (NS – number of model variants, number of simulations),
- calculation of model responses in NS cases,
- statistical analysis of responses.

The last step is done with the use of formulas known from statistics. For example, the estimators of mean \bar{u}_i and standard deviation s_{u_i} of the analysed response (e.g. displacement u_i) in an i -th degree of freedom of FEM model are [3, 6]:

$$\bar{u}_i = \frac{1}{NS} \sum_{j=1}^{NS} u_i(j) \quad (15)$$

$$s_{u_i} = \sqrt{\frac{\sum_{j=1}^{NS} [u_i(j)]^2 - NS \cdot (\bar{u}_i)^2}{NS-1}} \quad (16)$$

The accuracy of these estimations increases with an increasing number of simulations NS . On the other hand, large NS can lead to computational issues (especially for complex numerical models). For that reason, a number of sampling techniques, e.g. Latin Hyper Cube

Sampling (LHS) and Improved LHS, were developed. Their use may lead to the reduction of *NS* while maintaining an acceptable level of procedure’s accuracy.

2.4 Other probabilistic methods for analysis of FE model multiple responses

Not only the Monte Carlo Simulation is based on the multiple analyses of the deterministic FE model. A generalized variation of SFEM introduced by Kamiński [11] in a form of the Response Function Method (RFM) is one of them. Using such an approach, several real-life engineering structures were successfully analysed, e.g. [16, 17].

In this context, two other important methods should be mentioned here: Response Surface Method (RSM) [18] and Point Estimate Method (PEM) [19, 20]. Both rely on the approximation of the model's response and are widely used in civil engineering problems. However, it should be emphasized that these techniques are not classified as SFEM methods in [2] and [3].

3 SFEM applications in bridge analysis

The results of the author’s extensive review of the works regarding SFEM applications in the assessment of bridges are collected in Table 1. As demonstrated, the cited works contain bridge analysis issues related to, e. g.: bridge–vehicle interaction and weight in motion problems, dynamic response of pedestrian bridges, probabilistic seismic analysis, reliability and fatigue reliability issues, influence of wind excitations, and local failure of bridge components.

Table 1. Application of probabilistic FE methods in analysis of bridge structures.

method(s) used	object / element	aspect of analysis	ref.
Perturbation SFEM	cable-stayed bridge	prob. assessment of displacements and internal forces (random mat. properties)	[21]
Perturbation SFEM	three-span, box girder bridge	stochastic seismic analysis (random material properties)	[22]
Perturbation SFEM (RFM)	arch, pedestrian bridge	eigenfrequencies, displacements (uncertain dynamic load / deck stiffness / material properties)	[16]
SSFEM	single-span beam, road bridge	bridge–vehicle interaction (uncertain material parameters)	[23]
SSFEM	single-span beam, road bridge	bridge–vehicle interaction (uncertain dynamic loads, material density, elastic modulus, damping)	[24]
(K-L) expansion SFEM + MCS	single-span, laminated composite beam	bridge–vehicle interaction (randomness in material properties and moving loads)	[25]
(K-L) expansion SFEM	single-span beam, road bridge	bridge–vehicle interaction; moving force identification (random excitation and material properties)	[26]
FEM+MCS	cable of suspension bridge	probability distribution of cable failure (uncert. cable strength as 3D rand. field)	[27]
FEM+MCS (LHS)	steel, road bridge (single-span, two girders)	fatigue reliability assessment (uncertain vehicle loads in several aspects)	[28]

FEM+MCS (LHS)	three-span, suspension bridge	damping and flutter speed (structural and aerodynamic uncertainties)	[29]
FEM+MCS	three-span, suspension bridge (as in [29])	seismic analysis (uncertainties in seismic excitation)	[30]
FEM+MCS RSM	two PC box girders, four-span, road bridge	FE model validation (uncertain material properties, stiffness of support bearings and expansion joints, stiffness between two adjacent box-girders)	[31] [32]
FEM+MCS RSM	masonry arch bridge	stochastic response, namely stresses and displacements (random ground motion, uncertain material properties)	[33]
FEM+MCS RSM	large-span suspension bridge	Probabilistic free vibration and flutter analysis (uncertain material properties and stiffness)	[34]
FEM+ improved RSM (not SFEM)	large-span suspension bridge	risk assessment for the construction phases (random cross-section area of hangers and cables, elasticity of cables)	[35]
FEM+RSM (not SFEM)	arch bridge	system reliability assessment (uncertain section area, inertia, and resistant moments of the girders)	[36]

4 Computer implementation, feasibility issues

Although the range of SFEM applications in the probabilistic assessment of bridge structures seems wide, some limitations of this approach were identified, especially in regard to the level of computer implementation. There are not many widely available or commercial computer programs that implement the full SFEM procedure in the variant of SSFEM or Perturbation SFEM. Examples of such are SFESTA and SFEDYN. These programs were developed by Kleiber and Hien [4] in the Fortran® programming language. Both tools cover the scope of a deterministic, stochastic and sensitivity analysis with the use of the perturbation approach. The first one concerns the static analysis of 3D trusses and the second the dynamic analysis of 3D frames. Only one-dimensional elements can be used (no plate, shell or block elements can be used). These tools were further used and developed for the probabilistic computations of bridges by the authors of [21] and [22]. The presented results of a probabilistic seismic analysis of a cantilever bridge and a cable-stayed bridge indicate the high efficiency of perturbation SFEM in relation to MCS. However, to the author's knowledge, the aforementioned computer program, even after developments, does not have capabilities of using finite elements other than one-dimensional ones. Therefore, higher-class models (e2s2, e2s3, or e3s3) or the ones that use several element types are not applicable.

The spectral version of SFEM (SSFEM) is also used for rather simple structural models [23-26]. Despite the author's efforts in the literature review, no higher-class models of bridges were found to be evaluated by means of SSFEM. However, it should be emphasized that there are some open-source MATLAB® implementations of SSFEM available e.g. [37, 38], which can be further developed and modified in the future.

Advanced models with different types of finite elements – which are most commonly used in real-life bridge engineering – are still analysed by these SFEM variants which are based on multiple calculations of a deterministic model (which is often prepared in commercial FEM systems e.g. MIDAS®, LUSAS®, DIANA®, ARSAP®, SOFiSTiK®, SCIA®, ATENA®, CivilFEM®, RM Bridge®, ANSYS®, ABAQUS®, etc.) with later analysis of adequate responses. It is in particular MCS and RFM. Also, RSM (which is implemented to e.g. ANSYS®) and PEM are still very popular, and seem to be important alternatives to SFEM in many aspects of bridge infrastructure assessment computations.

5 Summary

Probabilistic tools are more and more often used for an analysis of bridge objects. It is not only about their traditional use, for example for the design of standards, but also about the analysis of specific bridge infrastructure objects in order to adequately represent sources of uncertainty. An important part of these methods is SFEM, which is implemented in three main versions: perturbation SFEM, SSFEM and FEM + MCS. The scope of applications of these methods in assessing bridges is wide. It concerns, among others, reliability, dynamic (mass in motion, bridge-vehicle interaction), seismic, wind load or static structural (including local effects analysis) analyses. Unfortunately, the implementation and use of the full SFEM procedure in a perturbation or spectral variant usually requires the creation of an individual computer code. Therefore, many of the collected examples refer to very simple FEM models. In the case of more sophisticated FE models and the usage of deterministic, commercial FEM programs well-known to bridge engineers, MSC and RFM seem to be a valuable solution. In many aspects, a strong alternative to the last two mentioned variants of SFEM are still RSM and PEM.

This paper was partially financed by a subsidy for the maintaining of research potential and the financing of young scientists (University of Bielsko-Biała, Poland, Faculty of Materials, Civil and Environmental Engineering)

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